Previously...

- We can determine a BFS ($m + n - 1$ lin indep vars)
- We have three ways of determining an initial BFS:
  - NW Corner Rule
  - Minimum Cost Method
  - Vogel's Approx Method (VAM)

We can formulate the transportation problem as an LP, and we can write its dual using u_i, v_j.

Goal today: Determine if a given BFS is optimal. If it is not, find a better BFS. (MODI- "Modified Distribution Method", or u-v method).

Example from Video 1 of 7.2: From the tableau, write down the LP and its dual.

	C1	C2	C3	
W 1	3	7	6	5
W 2	2	4	3	2
Demand	2	3	2	7

Original LP:

$$\begin{array}{rrrrrrrr}
\min\ w = & 3y_{11} & +7y_{12} & +6y_{13} & +2y_{21} & +4y_{22} & +3y_{23} & \\
\text{st} & y_{11} & +y_{12} & +y_{13} & & & & =5 \\
 & & & & y_{21} & +y_{22} & +y_{23} & =2 \\
\hline
 & y_{11} & & & +y_{21} & & & =2 \\
 & & y_{12} & & & +y_{22} & & =3 \\
 & & & y_{13} & & & +y_{23} & =2
\end{array}$$

Let u_i be dual var for supply, v_j be dual var for demand.

$$\max \ \ z = 5u_1 + 2u_2 + 2v_1 + 3v_2 + 2v_3$$

such that:

$$\begin{array}{lllll|lllll}
u_1 & +v_1 & & & \leq 3 & u_2 & +v_1 & & & \leq 2 \\
u_1 & & +v_2 & & \leq 7 & u_2 & & +v_2 & & \leq 4 \\
u_1 & & & +v_3 & \leq 6 & u_2 & & & +v_3 & \leq 3
\end{array}$$

where u_i, v_j are URS.

- By Complementary Slackness, if BV y_{ij} is > 0, the corresponding slack in the dual constraint is zero therefore:

  $$u_i + v_j = c_{ij}$$

  For the NBV, $u_i + v_j \leq c_{ij}$.
- We have an extra variable, set $u_1 = 0$ (This is a random choice).
- Solve for all other u_i, v_j belonging to BVs.
- For NBV's, compute $c_{ij} - (u_i + v_j)$ ("Row 0" in the LP)
- If these are all non-negative, the current solution is optimal.

	$v_1 =$	$v_2 =$	$v_3 =$	$v_4 =$	Supply
$u_1 =$	8 35	6	10	9	35
$u_2 =$	9 10	12 20	13 20	7	50
$u_3 =$	14	9	16 10	5 30	40
Demand	45	20	30	30	125

Current Value of $z = 1180$.

	$v_1 =$	$v_2 =$	$v_3 =$	$v_4 =$	Supply
$u_1 = 0$	8 35	6	10	9	35
$u_2 =$	9 10	12 20	13 20	7	50
$u_3 =$	14	9	16 10	5 30	40
Demand	45	20	30	30	125

Current Value of $z = 1180$.

MODI Example 1

	$v_1 = 8$	$v_2 =$	$v_3 =$	$v_4 =$	Supply
$u_1 = 0$	8 35	6	10	9	35
$u_2 =$	9 10	12 20	13 20	7	50
$u_3 =$	14	9	16 10	5 30	40
Demand	45	20	30	30	125

Current Value of $z = 1180$.

	$v_1 = 8$	$v_2 =$	$v_3 =$	$v_4 =$	Supply
$u_1 = 0$	8 35	6	10	9	35
$u_2 = 1$	9 10	12 20	13 20	7	50
$u_3 =$	14	9	16 10	5 30	40
Demand	45	20	30	30	125

Current Value of $z = 1180$.

MODI Example 1

	$v_1 = 8$	$v_2 = 11$	$v_3 =$	$v_4 =$	Supply
$u_1 = 0$	8 35	6	10	9	35
$u_2 = 1$	9 10	12 20	13 20	7	50
$u_3 =$	14	9	16 10	5 30	40
Demand	45	20	30	30	125

Current Value of $z = 1180$.

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 =$	Supply
$u_1 = 0$	8 35	6	10	9	35
$u_2 = 1$	9 10	12 20	13 20	7	50
$u_3 =$	14	9	16 10	5 30	40
Demand	45	20	30	30	125

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	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 =$	Supply
$u_1 = 0$	8 35	6	10	9	35
$u_2 = 1$	9 10	12 20	13 20	7	50
$u_3 = 4$	14	9	16 10	5 30	40
Demand	45	20	30	30	125

Current Value of $z = 1180$.

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = 1$	Supply
$u_1 = 0$	8 35	6	10	9	35
$u_2 = 1$	9 10	12 20	13 20	7	50
$u_3 = 4$	14	9	16 10	5 30	40
Demand	45	20	30	30	125

Current Value of $z = 1180$.

MODI Example 1

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = 1$	Supply
$u_1 = 0$	8 35	6	10	9	35
$u_2 = 1$	9 10	12 20	13 20	7	50
$u_3 = 4$	14 (2)	9	16 10	5 30	40
Demand	45	20	30	30	125

$$c_{ij} - (u_i + v_j) = 14 - (8 + 4) = 14 - 12 = 2$$

MODI Example 1

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = 1$	Supply
$u_1 = 0$	8 35	6 (-5)	10	9	35
$u_2 = 1$	9 10	12 20	13 20	7	50
$u_3 = 4$	14 (2)	9	16 10	5 30	40
Demand	45	20	30	30	125

Current Value of $z = 1180$.

MODI Example 1

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = 1$	Supply
$u_1 = 0$	8 35	6 (-5)	10	9	35
$u_2 = 1$	9 10	12 20	13 20	7	50
$u_3 = 4$	14 (2)	9 (-6)	16 10	5 30	40
Demand	45	20	30	30	125

Current Value of $z = 1180$.

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = 1$	Supply
$u_1 = 0$	8 35	6 (-5)	10 (-2)	9	35
$u_2 = 1$	9 10	12 20	13 20	7	50
$u_3 = 4$	14 (2)	9 (-6)	16 10	5 30	40
Demand	45	20	30	30	125

Current Value of $z = 1180$.

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = 1$	Supply
$u_1 = 0$	8 35	6 (-5)	10 (-2)	9 (8)	35
$u_2 = 1$	9 10	12 20	13 20	7 (5)	50
$u_3 = 4$	14 (2)	9 (-6)	16 10	5 30	40
Demand	45	20	30	30	125

Current Value of $z = 1180$.

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = 1$	Supply
$u_1 = 0$	8 35	6	10	9	35
$u_2 = 1$	9 10	12 $20 - \theta$	13 $20 + \theta$	7	50
$u_3 = 4$	14	9 θ	16 $10 - \theta$	5 30	40
Demand	45	20	30	30	125

Increase θ by as much as possible.

$$\begin{array}{c|c} 20-\theta & 20+\theta \\ \hline \theta & 10-\theta \end{array} \quad \rightarrow \quad \begin{array}{c|c} 10 & 30 \\ \hline 10 & \end{array}$$

The cell that becomes zero is removed from the set of basic variables.

MODI Example 1

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = 1$	Supply
$u_1 = 0$	8 35	6	10	9	35
$u_2 = 1$	9 10	12 10	13 30	7	50
$u_3 = 4$	14	9 10	16	5 30	40
Demand	45	20	30	30	125

New Value of $z = 1120$

MODI Example 1

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = 1$	Supply
$u_1 = 0$	8 35	6	10	9	35
$u_2 = 1$	9 10	12 10	13 30	7	50
$u_3 = 4$	14	9 10	16	5 30	40
Demand	45	20	30	30	125

Recompute $u's$ where necessary.
Note that v_2 doesn't change...
Compute u_3, then also v_4.

MODI Example 1

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = ??$	Supply
$u_1 = 0$	8 35	6	10	9	35
$u_2 = 1$	9 10	12 10	13 30	7	50
$u_3 = ??$	14	9 10	16	5 30	40
Demand	45	20	30	30	125

MODI Example 1

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = ??$	Supply
$u_1 = 0$	8 35	6	10	9	35
$u_2 = 1$	9 10	12 10	13 30	7	50
$u_3 = -2$	14	9 10	16	5 30	40
Demand	45	20	30	30	125

MODI Example 1

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = 7$	Supply
$u_1 = 0$	8 35	6	10	9	35
$u_2 = 1$	9 10	12 10	13 30	7	50
$u_3 = -2$	14	9 10	16	5 30	40
Demand	45	20	30	30	125

Recompute NBVs...

MODI Example 1

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = 7$	Supply
$u_1 = 0$	8 35	6 (-5)	10 (-2)	9 (2)	35
$u_2 = 1$	9 10	12 10	13 30	7 (-1)	50
$u_3 = -2$	14 (8)	9 10	16 (6)	5 30	40
Demand	45	20	30	30	125

MODI Example 1

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = 1$	Supply
$u_1 = 0$	8 $35 - \theta$	6 θ	10	9	35
$u_2 = 1$	9 $10 + \theta$	12 $10 - \theta$	13 30	7	50
$u_3 = 4$	14	9 10	16	5 30	40
Demand	45	20	30	30	125

We bring y_{12} into the set of BVs, and by using the loop, we'll remove y_{22} from the set.

MODI Example 1

	$v_1 = 8$	$v_2 = 11$	$v_3 = 12$	$v_4 = 1$	Supply
$u_1 = 0$	8 25	6 10	10	9	35
$u_2 = 1$	9 20	12	13 30	7	50
$u_3 = 4$	14	9 10	16	5 30	40
Demand	45	20	30	30	125

New value of $z = 1070$. Is it optimal?

	$v_1 = 8$	$v_2 =??$	$v_3 = 12$	$v_4 =??$	Supply
$u_1 = 0$	8 25	6 10	10	9	35
$u_2 = 1$	9 20	12	13 30	7	50
$u_3 =??$	14	9 10	16	5 30	40
Demand	45	20	30	30	125

Recalculating the dual...

MODI Example 1

	$v_1 = 8$	$v_2 = 6$	$v_3 = 12$	$v_4 = 2$	Supply
$u_1 = 0$	8 25	6 10	10	9	35
$u_2 = 1$	9 20	12	13 30	7	50
$u_3 = 3$	14	9 10	16	5 30	40
Demand	45	20	30	30	125

Next: Recalculate NBVs ("Row 0"):

	$v_1 = 8$	$v_2 = 6$	$v_3 = 12$	$v_4 = 2$	Supply
$u_1 = 0$	8 25	6 10	10 (-2)	9 (7)	35
$u_2 = 1$	9 20	12 (5)	13 30	7 (4)	50
$u_3 = 3$	14 (3)	9 10	16 (1)	5 30	40
Demand	45	20	30	30	125

Next: Bring in y_{13} and form a loop.

	$v_1 = 8$	$v_2 = 6$	$v_3 = 12$	$v_4 = 2$	Supply
$u_1 = 0$	8 $25 - \theta$	6 10	10 θ	9	35
$u_2 = 1$	9 $20 + \theta$	12	13 $30 - \theta$	7	50
$u_3 = 3$	14	9 10	16	5 30	40
Demand	45	20	30	30	125

Next: Take $\theta = 25$ and reset dual variables.

	$v_1 = 8$	$v_2 = 6$	$v_3 = 12$	$v_4 = 2$	Supply
$u_1 = 0$	8	6 10	10 25	9	35
$u_2 = 1$	9 45	12	13 5	7	50
$u_3 = 3$	14	9 10	16	5 30	40
Demand	45	20	30	30	125

New value of $z = 1020$.

Reset $u, v \ldots$

MODI Example 1

	$v_1 = ??$	$v_2 = 6$	$v_3 = ??$	$v_4 = 2$	Supply
$u_1 = 0$	8	6 10	10 25	9	35
$u_2 = ??$	9 45	12	13 5	7	50
$u_3 = 3$	14	9 10	16	5 30	40
Demand	45	20	30	30	125

MODI Example 1

	$v_1 = ??$	$v_2 = 6$	$v_3 = 10$	$v_4 = 2$	Supply
$u_1 = 0$	8	6 10	10 25	9	35
$u_2 = ??$	9 45	12	13 5	7	50
$u_3 = 3$	14	9 10	16	5 30	40
Demand	45	20	30	30	125

MODI Example 1

	$v_1 =??$	$v_2 = 6$	$v_3 = 10$	$v_4 = 2$	Supply
$u_1 = 0$	8	6 10	10 25	9	35
$u_2 = 3$	9 45	12	13 5	7	50
$u_3 = 3$	14	9 10	16	5 30	40
Demand	45	20	30	30	125

MODI Example 1

	$v_1 = 6$	$v_2 = 6$	$v_3 = 10$	$v_4 = 2$	Supply
$u_1 = 0$	8	6 10	10 25	9	35
$u_2 = 3$	9 45	12	13 5	7	50
$u_3 = 3$	14	9 10	16	5 30	40
Demand	45	20	30	30	125

Next: Recompute the Row 0 values in the NBV cells.

	$v_1 = 6$	$v_2 = 6$	$v_3 = 10$	$v_4 = 2$	Supply
$u_1 = 0$	8 (2)	6 10	10 25	9 (7)	35
$u_2 = 3$	9 45	12 (3)	13 5	7 (2)	50
$u_3 = 3$	14 (5)	9 10	16 (3)	5 30	40
Demand	45	20	30	30	125

Optimal.

In Class Example

Given the following tableau with BFS, compute the solution to the dual and determine if it is optimal. If not, say which cell should come into the basis.

	$v_1 =$	$v_2 =$	$v_3 =$	$v_4 =$	Supply
$u_1 =$	2 5	3	5	6	5
$u_2 =$	2 7	1 3	3	5	10
$u_3 =$	3	8 5	4 4	6 6	15
Demand	12	8	4	6	30

In Class Example

	$v_1 = 2$	$v_2 = 1$	$v_3 = -3$	$v_4 = -1$	Supply
$u_1 = 0$	2 5	3 (2)	5 (8)	6 (7)	5
$u_2 = 0$	2 7	1 3	3 (6)	5 (6)	10
$u_3 = 7$	3 (-6)	8 5	4 4	6 6	15
Demand	12	8	4	6	30

Bring in the $(3, 1)$ cell.

$$\begin{array}{c|c} 7-\theta & 3+\theta \\ \hline \theta & 5-\theta \end{array} \quad \rightarrow \quad \begin{array}{c|c} 2 & 8 \\ \hline 5 & \end{array}$$

Now enter these variables, re-compute the dual and the Row 0 values.

	$v_1 = 2$	$v_2 = 1$	$v_3 = 3$	$v_4 = 5$	Supply
$u_1 = 0$	2 5	3 (2)	5 (2)	6 (2)	5
$u_2 = 0$	2 2	1 8	3 (0)	5 (0)	10
$u_3 = 1$	3 5	8 (6)	4 4	6 6	15
Demand	12	8	4	6	30

This is optimal.

Degeneracy

	$v_1 =$	$v_2 =$	$v_3 =$	$v_4 =$	Supply
$u_1 =$	4 20	6 20	8	8	40
$u_2 =$	6	8 10	6 50	7	60
$u_3 =$	5	7	6	8 50	50
Demand	20	30	50	50	150

We cannot solve for the dual...

We must decide on which variable will be basic. (Put ϵ in that cell)

We do not want a loop!

Degeneracy

	$v_1 =$	$v_2 =$	$v_3 =$	$v_4 =$	Supply
$u_1 =$	4 20	6 20	8 No	8	40
$u_2 =$	6 No	8 10	6 50	7	60
$u_3 =$	5	7	6	8 50	50
Demand	20	30	50	50	150

Degeneracy

	$v_1 =$	$v_2 =$	$v_3 =$	$v_4 =$	Supply
$u_1 =$	4 20	6 20	8	8	40
$u_2 =$	6	8 10	6 50	7	60
$u_3 =$	5	7	6 ϵ	8 50	50
Demand	20	30	50	50	150

Degeneracy

Now we can fill in the dual:

	$v_1 = 4$	$v_2 = 6$	$v_3 = 4$	$v_4 = 6$	Supply
$u_1 = 0$	4 20	6 20	8	8	40
$u_2 = 2$	6	8 10	6 50	7	60
$u_3 = 2$	5	7	6 ϵ	8 50	50
Demand	20	30	50	50	150

Degeneracy

Check for optimality:

	$v_1 = 4$	$v_2 = 6$	$v_3 = 4$	$v_4 = 6$	Supply
$u_1 = 0$	4 20	6 20	8 (4)	8 (2)	40
$u_2 = 2$	6 (0)	8 10	6 50	7 (-1)	60
$u_3 = 2$	5 (-1)	7 (-1)	6 ϵ	8 50	50
Demand	20	30	50	50	150

Degeneracy

We'll choose to bring in y_{32}, which gives us the loop:

	$v_1 = 4$	$v_2 = 6$	$v_3 = 4$	$v_4 = 6$	Supply
$u_1 = 0$	4 20	6 20	8 (4)	8 (2)	40
$u_2 = 2$	6 (0)	8 10	6 50	7 (-1)	60
$u_3 = 2$	5 (-1)	7 (-1)	6 ϵ	8 50	50
Demand	20	30	50	50	150

Degeneracy

$$\begin{array}{r|r} 10-\theta & 50+\theta \\ \hline \theta & \epsilon-\theta \end{array} \quad \Rightarrow \quad \begin{array}{r|l} 10 & 50 \\ \hline \epsilon & \end{array}$$

This is a common occurrence, and the reason we use ϵ and not 0.

This could change our computation of the dual...

Degeneracy, continued

Here are our new values for the dual...

	$v_1 = 4$	$v_2 = 6$	$v_3 = 4$	$v_4 = 7$	Supply
$u_1 = 0$	4 20	6 20	8 (4)	8 (1)	40
$u_2 = 2$	6 (0)	8 10	6 50	7 (-2)	60
$u_3 = 1$	5 (0)	7 ϵ	6 (1)	8 50	50
Demand	20	30	50	50	150

Degeneracy, continued

Bring y_{24} into the basis, and we have the loop below:

	$v_1 = 4$	$v_2 = 6$	$v_3 = 4$	$v_4 = 7$	Supply
$u_1 = 0$	4 20	6 20	8	8	40
$u_2 = 2$	6	8 $10 - \theta$	6 50	7 θ	60
$u_3 = 1$	5	7 $\epsilon + \theta$	6	8 $50 - \theta$	50
Demand	20	30	50	50	150

With $\theta = 10$, we will remove the degeneracy!

Continuing...

Here is the new tableau with the new dual values computed. We only show negative values of $c_{ij} - (u_i + v_j)$.

	$v_1 = 4$	$v_2 = 6$	$v_3 = 6$	$v_4 = 7$	Supply
$u_1 = 0$	4 20	6 20	8	8	40
$u_2 = 0$	6	8	6 50	7 10	60
$u_3 = 1$	5	7 10	6 (-1)	8 40	50
Demand	20	30	50	50	150

Continuing...

Bring in y_{33}, and we have an optimal tableau:

	$v_1 = 4$	$v_2 = 6$	$v_3 = 5$	$v_4 = 6$	Supply
$u_1 = 0$	4 20	6 20	8	8	40
$u_2 = 1$	6	8	6 10	7 50	60
$u_3 = 1$	5	7 10	6 40	8	50
Demand	20	30	50	50	150

Next up: Sensitivity Analysis