Sensitivity Analysis

In this case, we'll switch to the transportation problem in the text. The optimal tableau was found to be:

	$v_1 = 6$	$v_2 = 6$	$v_3 = 10$	$v_4 = 2$	Supply
$u_1 = 0$	[8]	[6] 10	[10] 25	[9]	35
$u_2 = 3$	[9] 45	[12]	[13] 5	[7]	50
$u_3 = 3$	[14]	[9] 10	[16]	[5] 30	40
Demand	45	20	30	30	125

There are several changes we'll look at:

- Change the cost, c_{ij} for a NBV.
- Change the cost for a basic variable.
- Change supply s_i and demand d_j by Δ (must do both to stay balanced).

Computations for Sensitivity Analysis

We'll use the book's problem as our example. Here we go with some details.

1. Change a cost c_{ij} for a NBV.

   In this situation, the change is completely localized to the (i, j) cell. To keep the current basis optimal, we only need:

   $$(c_{ij} + \Delta) - (u_i + v_j) = (c_{ij} - (u_i + v_j)) + \Delta > 0$$

   For example, changing the c_{11} cost from 8 to $8 + \Delta$ results in:

   $$(8 - (6 + 0)) + \Delta > 0 \quad \Rightarrow \quad \Delta > -2$$

   As an extra example, suppose we change by $\Delta = -3$. Compute the new basic solution.

   SOLUTION: Changing the cost to 5 will make the "Row 0" value negative, meaning we now want to pivot into x_{11}. Doing that we get the loop:

   $$\begin{array}{r|c|r} \theta & (10) & 25-\theta \\ \hline 45-\theta & & 5+\theta \end{array} \quad \Rightarrow \quad \begin{array}{c|c|c} 25 & 10 & \\ \hline 20 & & 30 \end{array}$$

Recalculating "Row 0" values:

	$v_1 = 5$	$v_2 = 6$	$v_3 = 9$	$v_4 = 2$	Supply
	5	6	10	9	
$u_1 = 0$	25	10	+	+	35
	9	12	13	7	
$u_2 = 4$	20	+	30	+	50
	14	9	16	5	
$u_3 = 3$	+	10	+	30	40
Demand	45	20	30	30	125

And we see that the current tableau is again optimal.

2. Changing the cost of a BV.

   In this situation, adding Δ will have more of an effect, since these costs are used in calculating the u, v.

   For example, let's change c_{13} from 10 to $10 + \Delta$ and track what happens:

	$v_1 = 6 + \Delta$	$v_2 = 6$	$v_3 = 10 + \Delta$	$v_4 = 2$	Supply
	8	6	$10 + \Delta$	9	
$u_1 = 0$	$(2 - \Delta)$	10	25	(7)	35
	9	12	13	7	
$u_2 = 3 - \Delta$	45	$(3 + \Delta)$	5	$(2 + \Delta)$	50
	14	9	16	5	
$u_3 = 3$	$(5 - \Delta)$	10	$(3 - \Delta)$	30	40
Demand	45	20	30	30	125

   We have 5 inequalities to check:

$$2 - \Delta > 0, \quad 5 - \Delta > 0, \quad 3 + \Delta > 0, \quad 3 - \Delta > 0, \quad 2 + \Delta > 0$$

   All are satisfied (do them on a number line) for $-2 < \Delta < 2$.

3. Change s_i, d_j when x_{ij} is basic:

   This means that the solution is simply increased by the common amount. For example, if s_1, d_2 were increased by 2, then we increase x_{12} from 10 to 12. This increases the cost by $2 \times c_{12} = 2 \times 6 = 12$.

4. Change s_i, d_j when x_{ij} is non-basic: In this case, we need to "absorb" the change in the loop that is created by temporarily increasing x_{ij} from 0.

   For example, suppose supply 1, demand 4 is increased to Δ. What is the corresponding change to the basic variables?

	$v_1 = 6$	$v_2 = 6$	$v_3 = 10$	$v_4 = 2$	Supply
$u_1 = 0$	8	6; 10	10; 25	9; Δ	$35 + \Delta$
$u_2 = 3$	9; 45	12	13; 5	7	50
$u_3 = 3$	14	9; 10	16	5; 30	40
Demand	45	20	30	30+Δ	125

This creates the loop:

$10 + \Delta$	(25)	$\Delta - \Delta$
$10 - \Delta$		$30 + \Delta$

And because the costs and shadow prices did not change, we are still optimal:

	$v_1 = 6$	$v_2 = 6$	$v_3 = 10$	$v_4 = 2$	Supply
$u_1 = 0$	8	6; 10+Δ	10; 25	9	$35 + \Delta$
$u_2 = 3$	9; 45	12	13; 5	7	50
$u_3 = 3$	14	9; 10$-\Delta$	16	5; $30 + \Delta$	40
Demand	45	20	30	30+Δ	125

With the change in $z = z_{\text{old}} + 6\Delta + 5\Delta - 9\Delta = z_{\text{old}} + 2\Delta$