Review Questions, Exam 1, Ops Research

1. There were 4 assumptions when constructing a linear program. List them (Hint: One was "Proportionality").

2. What are the four possible outcomes when solving a linear program? Hint: The first is that there is a unique solution to the LP.

3. The following are to be sure you understand the process of constructing a linear program:

   (a) Exercise 2, 31 Chapter 3 review (Be sure you can solve an LP graphically)

   (b) Exercise 6, 18 Chapter 3 review (A ton is 2000 lbs)

   (c) Exercise 22, Chapter 3 review. Hint: Consider using a triple index on your variables.

   (d) Exercise 47 in Chapter 3 review.

   (e) Exercise 12 in Chapter 4 review (set up the LP only for "set 1" only- do NOT solve).

   (f) Exercise 17 in Chapter 4 review.

4. Convert the following LP to one in standard form. Write the result in matrix-vector form, giving $\mathbf{x}$, $\mathbf{c}$, $A, \mathbf{b}$ (from our formulation).

$$\begin{aligned} \min z = \ & 3x - 4y + 2z \\ \text{st} \ & 2x - 4y \geq 4 \\ & x + z \geq -5 \\ & y + z \leq 1 \\ & x + y + z = 3 \end{aligned}$$

   with $x \geq 0, y$ is URS, $z \geq 0$.

5. Suppose the BFS for an optimal tableau is degenerate and a NBV in Row 0 has a zero coefficient. Show by example that either of the following could occur:

   - The LP has more than one optimal solution.
   - The LP has a unique optimal solution.

6. Consider again the "Wyndoor" company example we looked at in class:

$$\begin{aligned} \min z = \ & 3x_1 + 5x_2 \\ \text{st} \ & x_1 \leq 4 \\ & 2x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 18 \end{aligned}$$

   with x_1, x_2 both non-negative.

(a) Rewrite so that it is in standard form.

(b) Let s_1, s_2, s_3 be the extra variables introduced in the last answer. Is the following a basic solution? Is it a basic feasible solution?

$$x_1 = 0, x_2 = 6, s_1 = 4, s_2 = 0, s_3 = 6$$

Which variables are BV, and which are NBV?

(c) Find the basic feasible solution obtained by taking s_1, s_3 as the non-basic variables.

7. Draw the feasible set corresponding to the following inequalities:

$$x_1 + x_2 \le 6, \quad x_1 - x_2 \le 2 \quad x_1 \le 3, \quad x_2 \le 6$$

with x_1, x_2 non-negative.

(a) Find the set of extreme points.

(b) Write the vector $[1, 1]^T$ as a convex combination of the extreme points.

(c) True or False: The extreme points of the region can be found by making exactly two of the constraints binding, then solve.

(d) If the objective function is to maximize $2x_1 + x_2$, then (a) how might I change that into a minimization problem, and (b) solve it.

8. Consider the unbounded feasible region defined by

$$x_1 - 2x_2 \le 4, \qquad -x_1 + x_2 \le 3$$

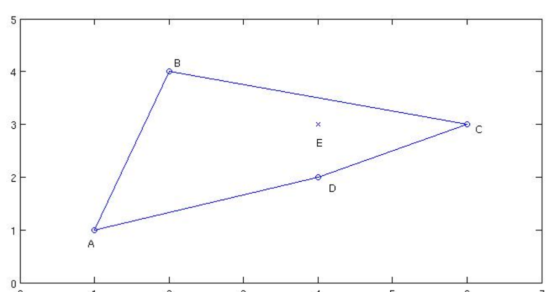
with x_1, x_2 non-negative. Consider the vector $\mathbf{p} = [5, 2]$.

(a) Show that $\mathbf{p}$ is in the feasible region.

(b) Set up the system you would solve in order to write $\mathbf{p}$ in the form given in Theorem 2 above (provide a specific vector $\mathbf{d}$).

9. Consider the figure with points $A(1, 1), B(1, 4)$ and $C(6, 3)$, $D(4, 2)$ and $E(4, 3)$.

- Write the point E as a convex combination of points A, B and C.
- Can E be written as a convex combination of A, B and D? If so, construct it.
- Can A be written as a *linear* combination of A, B and D? If so, construct it.



10. Finish the definition: Two basic feasible solutions are said to be **adjacent** if:

11. Let $\mathbf{d}$ be a direction of unboundedness. Using the *definition*, prove that this means that $r\mathbf{d}$ is also a direction of unboundedness, for any constant $r \geq 0$.

12. If C is a convex set, then $\mathbf{d} \neq 0$ is a direction of unboundedness for C iff $\mathbf{x} + d \in C$ for all $\mathbf{x} \in C$ (Use the *definition* of unboundedness).

13. For an LP in standard form (see above), prove that the vector $\mathbf{d}$ is a direction of unboundedness iff $A\mathbf{d} = 0$ and $\mathbf{d} \geq 0$.

14. Show that the set of optimal solutions to an LP (assume in standard form) is convex.

15. Let a feasible region be defined by the system of inequalities below:

$$\begin{array}{rl} -x_1 + 2x_2 & \leq 6 \\ -x_1 + x_2 & \leq 2 \\ x_2 & \geq 1 \\ x_1, x_2 \geq 0 & \end{array}$$

    The point $(4, 3)$ is in the feasible region. Find vectors $\mathbf{d}$ and $\mathbf{b}_1, \cdots \mathbf{b}_k$ and constants σ_i so that the Representation Theorem is satisfied (NOTE: Your vector $\mathbf{x}$ from that theorem is more than two dimensional).

16. Let a feasible region be defined by the system of inequalities below:

$$\begin{array}{rl} -x_1 + x_2 & \leq 2 \\ x_1 - x_2 & \leq 1 \\ x_1 + x_2 & \leq 5 \\ x_1, x_2 \geq 0 & \end{array}$$

    The point $(2, 2)$ is in the feasible region. Find vectors $\mathbf{d}$ and $\mathbf{b}_1, \cdots \mathbf{b}_k$ and constants σ_i so that the Representation Theorem is satisfied (NOTE: Your vector $\mathbf{x}$ from that theorem is more than two dimensional).

17. These two proofs go together- the first one you can "hand waive" through.

    (a) Suppose that $\lambda_1 \leq \lambda_2 \leq \cdots \lambda_n$, and let $\sigma_1, \cdots, \sigma_n$ be non-negative constants so that $\displaystyle\sum_{i=1}^{n} \sigma_i = 1$. Show that

$$\lambda_1 \leq \sigma_1\lambda_1 + \sigma_2\lambda_2 + \cdots \sigma_n\lambda_n \leq \lambda_n$$

    (b) Show that, if $\mathbf{x}$ is in the convex hull of vectors $\mathbf{b}_1, \cdots \mathbf{b}_k$, then for any constant vector $\mathbf{c}$,

$$\mathbf{c}^T\mathbf{x} \leq \max_i \left\{\mathbf{c}^T\mathbf{b}_i\right\}$$

18. True or False, and explain: The Simplex Method will always choose a basic feasible solution that is **adjacent** to the current BFS.

19. Given the current tableau (with variables labeled above the respective columns), answer the questions below.

x_1	x_2	s_1	s_2	rhs
0	-1	0	2	24
0	$1/3$	1	$-1/3$	1
1	$2/3$	0	$1/3$	4

   (a) Is the tableau optimal (and did your answer depend on whether we are maximizing or minimizing)? For the remaining questions, you may assume we are maximizing.

   (b) Give the current BFS.

   (c) Directly from the tableau, can I increase x_2 from 0 to 1 and remain feasible? Can I increase it to 4?

   (d) If x_2 is increased from 0 to 1, compute the new value of z, x_1, s_1 (assuming s_2 stays zero).

   (e) Write the objective function and all variables in terms of the non-basic (or free) variables, and then put them in vector form.

20. Solve first using big-M, then repeat using the two-phase method.

$$
\begin{array}{rrrl}
\max \quad z = & 5x_1 & -x_2 & \\
\text{st} \quad & 2x_1 & +x_2 & = 6 \\
 & x_1 & +x_2 & \leq 4 \\
 & & x_2 & \leq 3 \\
 & x_1, & x_2 & \geq 0
\end{array}
$$

21. Using the big-M method on a maximization problem, I got the following tableau:

	x_1	x_2	x_3	s_1	e_1	e_2	a_1	a_2	rhs
	$-1/2+2M$	$-5/2+M$	M	$1/2+M$	M	M	0	0	$2-3M$
x_3	$1/2$	$1/2$	1	$1/2$	0	0	0	0	2
a_1	$-3/2$	$-1/2$	0	$-1/2$	-1	0	1	0	2
a_2	$-1/2$	$-1/2$	0	$-1/2$	0	-1	0	1	1

Should I stop or should I go? If I stop, what should I conclude?

22. Here's a tableau that we've obtained from using the Simplex Method. Answer the questions below about it.

x_1	x_2	x_3	s_1	s_2	s_3	rhs
1	0	0	0	3	-2	50
4	0	0	1	-1	0	5
1	1	0	0	1	-2	10
0	0	1	0	1	-1	15

(a) Is this tableau terminal (has the Simplex Method stopped)? If so, interpret the solution shown. If not, continue until you stop.

(b) Write down the system of equations that this tableau represents (be sure to write BVs in terms of NBVs).

(c) Given the tableau shown, the current basic variables are s_1, x_2, x_3. Is it possible that the **previous** set of basic variables were: s_1, s_2, x_3? To see, compute the previous Row 0. (Hint: You want to replace or substitute x_2 with s_2 as basic).