Solutions to Review Questions, Exam 1

1. What are the four assumptions for a linear program?

   - Proportionality: The contribution to both the objective function and the constraints from a decision variable needs to be proportional- In the form of a constant times the variable. That means, no discounts for selling more, no fixed costs, etc.
   - Additive: The contribution to both the objective function and the constraints from variables x_1, x_2 (or more) must be additive, or $c_1x_1 + c_2x_2$. For example, you cannot have the contribution be a product of variables.

     The first two items ensures that the linear program is "linear" in both the objective function and constraints.
   - Divisibility: We assume decision variables take on real numbers (not only integers, for example).
   - Predictability: All constants are known- We do not use a probability distribution for costs, for example.

2. What are the four possible outcomes when solving a linear program? Hint: The first is that there is a unique solution to the LP.

   SOLUTION:

   - No solution - The feasible set is empty.
   - A unique solution (either with or without an unbounded feasible set).
   - An unbounded solution - The feasible set is unbounded.
   - An infinite number of solutions - Either by an unbounded set or the isoprofit lines are coincident with a boundary at the optimum.

3. The following are to be sure you understand the process of constructing a linear program:

   (a) Exercise 2, 31 Chapter 3 review
   SOLUTIONS: Be sure you can do these graphically.
   - Exercise 2: The optimal value is 69/7, where we have 36/7 chocolate cake and 66/7 vanilla (yes, the "divisibility" assumption is violated here).
   - Exercise 31: The LP is unbounded (no solution).

   (b) Exercise 6, 18 Chapter 3 review (A ton is 2000 lbs)
   SOLUTIONS: Be sure you introduce your variables!

   6. Let x_1 be the pounds of Alloy 1 used to produce one ton of steel and x_2 be the pounds of Alloy 2. Then the objective function is:

   $$\min z = 190/2000x_1 + 200/2000x_2$$

   With:
   - Carbon constraints:

   $$0.03x_1 + 0.04x_2 \geq (0.032)(2000) \qquad 0.03x_1 + 0.04x_2 \leq (0.035)(2000)$$

- Silicon:

$$0.02x_1 + 0.025x_2 \geq (0.018)(2000) \qquad 0.02x_1 + 0.025x_2 \leq (0.025)(2000)$$

- Lastly, nickel:

$$0.01x_1 + 0.015x_2 \geq 18 \qquad 0.01x_1 + 0.015x_2 \leq 24$$

- Tensile strength:

$$\frac{42,000x_1 + 50,000x_2}{2,000} \geq 45,000$$

- Relationship between variables: $x_1 + x_2 = 2000$
- Non-negative: $x_{1,2} \geq 0$.

18. An interesting application of "blending"- The solution is attached on the last page (a little lengthy to get it set up).

(c) Exercise 22, Chapter 3 review. Hint: Consider using a triple index on your variables.

SOLUTION:

Let x_{ijk} is the units of product 1, machine i, month j, for sale in month k.

Let y_{ijk} is the units of product 2, machine i, month j, for sale in month k.

In order to simplify things, we note some quantities that are useful:

- Amount of Product 1 for sale in Month 1: $x_{111} + x_{211}$
- Amount of Product 2 for sale in Month 1: $y_{111} + y_{211}$
- Amount of Product 1 for sale in Month 2: $x_{112} + x_{212} + x_{122} + x_{222}$
- Amount of Product 2 for sale in Month 2: $y_{112} + y_{212} + y_{122} + y_{222}$

Then we have the objective function to maximize:

$55(x_{111}+x_{211})+12(x_{112}+x_{212}+x_{122}+x_{222})+65(y_{111}+y_{211})+32(y_{112}+y_{212}+y_{122}+y_{222})$

For constraints, here are Machine 1 hour constraints (for Months 1, 2):

$$4(x_{111} + x_{112}) + 7(y_{111} + y_{112}) \leq 500 \qquad 4x_{122} + 7y_{122} \leq 500$$

Similarly, for Machine 2:

$$3(x_{211} + x_{212}) + 4(y_{211} + y_{212}) \leq 500 \qquad 3x_{222} + 4y_{122} \leq 500$$

Sales constraints (Month 1, then Month 2):

$$x_{111} + x_{211} \leq 100 \qquad y_{111} + y_{211} \leq 140$$

$$x_{112} + x_{212} + x_{122} + x_{222} \leq 190 \qquad y_{112} + y_{212} + y_{122} + y_{222} \leq 130$$

Also, all variables are non-negative.

(d) Exercise 47, Ch 3 Review: To get started, we might take

$$x_{ij} = \text{ Number of workers getting } i \text{ and } j \text{ off}, i < j$$

where Sunday is Day 1, Monday is Day 2, etc. We want to optimize the number of workers having consecutive days off:

$$\max z = x_{12} + x_{17} + x_{23} + x_{34} + x_{45} + x_{56} + x_{67}$$

subject to the day constraints (see the back of the text).

(e) Exercise 12, Ch 4 Review (Set up the LP just for "set 1", and do not solve)

SOLUTION: First the variables- Let x_i be the number of product i produced. Going off Table 70, we see the resource constraints (before anything extra is paid for):

$$\begin{aligned} 1.5x_1 + 3x_2 + 2x_3 &\leq 900 && \text{Labor} \\ 2x_1 + 3x_2 + 4x_3 &\leq 1600 && \text{Lumber} \\ 3x_1 + 2x_2 + 2x_3 &\leq 1550 && \text{Paint} \end{aligned}$$

Let s_i be the left over amount of labor, lumber, or paint (respectively) we'll have and e_i be the amount of labor, lumber, or paint that is beyond our available resources that we have to purchase. Then the constraints change, and rearranging the constraints to match our priorities,

$$\begin{aligned} 1.5x_1 + 3x_2 + 2x_3 + s_1 - e_1 &= 900 && \text{Labor} \\ 3x_1 + 2x_2 + 2x_3 + s_3 - e_3 &= 1550 && \text{Paint} \\ 2x_1 + 3x_2 + 4x_3 + s_2 - e_2 &= 1600 && \text{Lumber} \end{aligned}$$

Now that these excess variables are available, we can write our "profit" constraint (we'll use s_4, e_4 for our new constraint)

$$(26-10)x_1 + (28-6)x_2 + (31-7)x_3 - 6e_1 - 3e_2 - 2e_3 + s_4 - e_4 = 10500$$

Now for our priorities: Using P_i, we initially want to maximize P_1e_4 to get at least 10500 profit, then minimize additional labor, P_2e_1, then minimize additional paint, P_3e_3, then minimize additional lumber, P_4e_2.

(f) Exercise 17, Ch 4 Review. Given the table (Table 74) below, and assuming we started with a maximization problem,

x_1	x_2	x_3	x_4	x_5	rhs
$-c$	2	0	0	0	10
-1	a_1	1	0	0	4
a_2	-4	0	1	0	1
a_3	3	0	0	1	b

i. What conditions on the constants would make the current solution optimal?
SOLUTION: Make $c \leq 0$ and $b \geq 0$.

ii. What conditions on the constants would make the current solution optimal, with alternate optimal solutions?
SOLUTION: Make $c = 0$ and $b \geq 0$. Furthermore, we would need to be able to pivot into Column 1, so that one or both of a_2, a_3 would need to be greater than 0 so that we can successfully complete the pivot.

iii. What conditions on the constants would make the current tableau represent an unbounded solution (assume $b \geq 0$).
SOLUTION: If $c > 0$ and we cannot pivot (meaning $a_2 \leq 0$ and $a_3 \leq 0$), then the LP is unbounded.

4. Convert the following LP to one in standard form. Write the result in matrix-vector form, giving $\mathbf{x}$, $\mathbf{c}, A, \mathbf{b}$ (from our formulation).

$$\begin{aligned} \min z = \ & 3x - 4y + 2z \\ \text{st} \ & 2x - 4y \geq 4 \\ & x + z \geq -5 \\ & y + z \leq 1 \\ & x + y + z = 3 \end{aligned}$$

with $x \geq 0, y$ is URS, $z \geq 0$.

SOLUTUION: Let $\mathbf{x} = [x, y^+, y^-, z, e_1, s_1, s_2]^T$. Then

$$\mathbf{c} = [3, -4, 4, 2, 0, 0, 0]^T \qquad A = \begin{bmatrix} 2 & -4 & 4 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 1 \\ 1 & 1 & -1 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 1 \\ 3 \end{bmatrix}$$

5. Suppose the BFS for an optimal tableau is degenerate and a NBV in Row 0 has a zero coefficient. Show by example that either of the following could occur:

- The LP has more than one optimal solution.

  SOLUTION: Multiple ways of answering. In the tableau below, we can still pivot into Column 3 to get a new solution.

  $$\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 5 \\ \hline 1 & 0 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 & 1 \end{array}$$

- The LP has a unique optimal solution.

  SOLUTION: Changing the tableau above, we just need to make sure we cannot pivot into Col 3. For example,

  $$\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 5 \\ \hline 1 & 0 & -1 & 1 & 3 \\ 0 & 1 & -2 & 1 & 1 \end{array}$$

6. Consider again the "Wyndoor" company example we looked at in class:

$$\begin{aligned} \min z = \ & 3x_1 + 5x_2 \\ \text{st} \ & x_1 \leq 4 \\ & 2x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 18 \end{aligned}$$

with x_1, x_2 both non-negative.

(a) Rewrite so that it is in standard form.

SOLUTION:

Define the extra variables x_3, x_4, x_5.

Using the extra variables in order, the constraints become:

$$\begin{array}{rrrrrl} x_1+ & & x_3 & & & = 4 \\ & 2x_2+ & & x_4 & & = 12 \\ 3x_1+ & 2x_2+ & & & x_5 & = 18 \end{array}$$

And from this, it is easy to read off the coefficient matrix A.

(b) Let s_1, s_2, s_3 be the extra variables introduced in the last answer. Is the following a basic solution? Is it a basic feasible solution?

$$x_1 = 0, x_2 = 6, s_1 = 4, s_2 = 0, s_3 = 6$$

Which variables are BV, and which are NBV?

SOLUTION: The matrix A has rank 3. If the solution has $n - m = 5 - 3 = 2$ zeros (and it is a solution), then it is a basic solution: Yes, this is a basic solution. It is also a basic *feasible* solution since every entry of the basic solution is non-negative. The variables x_2, x_3 and x_5 are the basic variables (BV) and the variables x_1 and x_4 are NBV.

(c) Find the basic feasible solution obtained by taking s_1, s_3 as the non-basic variables.

In this case, we can row reduce the augmented matrix (remove columns 3 and 5 from the original):

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 2 & 1 & 12 \\ 3 & 2 & 0 & 18 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

In this case, we have the (full) solution:

$$x_1 = 4, \quad x_2 = 3, \quad x_3 = 0, \quad x_4 = 6, \quad x_5 = 0$$

7. Draw the feasible set corresponding to the following inequalities:

$$x_1 + x_2 \leq 6, \quad x_1 - x_2 \leq 2 \quad x_1 \leq 3, \quad x_2 \leq 6$$

with x_1, x_2 non-negative.

(a) Find the set of extreme points.

SOLUTION: $(0, 0)$, $(0, 6)$, $(2, 0)$, $(3, 3)$, $(3, 1)$.

(b) Write the vector $[1, 1]^T$ as a convex combination of the extreme points.

SOLUTION: Since I get to choose, let's make it easy:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 \\ 3 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 6 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Or, a little more complex:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 0 \\ 6 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 3 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

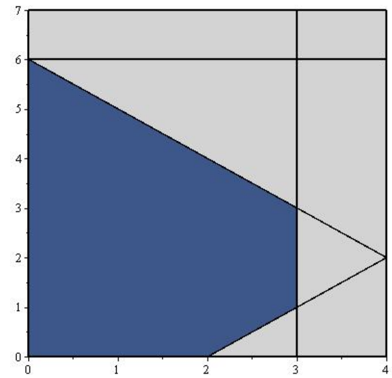


Figure 1: Figure for Question 10

(c) True or False: The extreme points of the region can be found by making exactly two of the constraints binding, then solve.

SOLUTION: If we follow this recipe, we will get extreme points, but we'll also get non-feasible points (for example, the point $(3, 6)$). Therefore, FALSE.

(d) If the objective function is to maximize $2x_1 + x_2$, then (a) how might I change that into a minimization problem, and (b) solve it.

SOLUTION: For part (a), we convert it by minimizing $-z$, or $\min -2x_1 - x_2$. For part (b), solve it graphically to get that the maximum occurs at $(3, 3)$ and the maximum is 9.

8. Consider the unbounded feasible region defined by

$$x_1 - 2x_2 \leq 4, \qquad -x_1 + x_2 \leq 3$$

with x_1, x_2 non-negative. Consider the vector $\mathbf{p} = [5, 2]$.

(a) Show that $\mathbf{p}$ is in the feasible region.

SOLUTION: Substitute the values into the constraints to see that they are both valid.

(b) Set up the system you would solve in order to write $\mathbf{p}$ in the form given in Theorem 2 (provide a specific vector $\mathbf{d}$).

NOTE: Sorry about the vagueness of the question... It is not clear what is being asked here. Go ahead and write the representation using two dimensions, then try to convert those to the four dimensional representation by writing the appropriate equations.

SOLUTION: Directions of unboundedness can have "slopes" between 1/2 and 1, so we could choose $\mathbf{d} = [1, 1]^T$ (so the vector has slope 1). We can construct the line through $(5, 2)$ with slope 1 (which is $y = x - 3$) and see where it connects with the convex hull of the extreme points. We note that $y = x - 3$ intersects the line segment between $(0, 0)$ and $(4, 0)$ at the point $(3, 0)$, so we can now write

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

where we'll take $(2,2)$ as the direction of unboundedness. As we noted before, $(3,0)$ is a convex combination of $(0,0)$ and $(4,0)$:

$$(3,0) = \frac{1}{4}(0,0) + \frac{3}{4}(4,0)$$

Therefore, in column vector format (you can do it either way):

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{3}{4}\begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Now, to put these vectors in the right dimension for Theorem 2, we need to write our constraints as $A\mathbf{x} = \mathbf{b}$, or

$$\begin{aligned} x_1 - 2x_2 + s_1 &= 4 \\ -x_1 + x_2 + s_2 &= 3 \end{aligned}$$

The points $(0,0)$, $(4,0)$, and $(5,2)$ are all feasible, so we use the basic equations above to solve for s_1, s_2:

$$\begin{bmatrix} 0 \\ 0 \\ 4 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 0 \\ 0 \\ 7 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 2 \\ 5 \\ 6 \end{bmatrix}$$

For the direction of unboundedness $(2,2)$, we set the equations to zero:

$$\begin{aligned} x_1 - 2x_2 + s_1 &= 0 \\ -x_1 + x_2 + s_2 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} 2 \\ 2 \\ 4 \\ 0 \end{bmatrix}$$

And we can verify these:

$$\begin{bmatrix} 5 \\ 2 \\ 5 \\ 6 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 \\ 0 \\ 4 \\ 3 \end{bmatrix} + \frac{3}{4}\begin{bmatrix} 4 \\ 0 \\ 0 \\ 7 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 4 \\ 0 \end{bmatrix}$$

9. Consider Figure 2, with points $A(1,1), B(1,4)$ and $C(6,3)$, $D(4,2)$ and $E(4,3)$.

- Write the point E as a convex combination of points A, B and C.

  SOLUTION: First we'll find the point of intersection between line $\overline{AE}$ and $\overline{BC}$. Call it E'. We found it to be $E'\left(\frac{58}{13}, \frac{43}{13}\right)$ (Sorry about the fractions!).

  By the time we're done, you should have:

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \frac{2}{15}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{13}{15}\left(\frac{4}{13}\begin{bmatrix} 1 \\ 4 \end{bmatrix} + \frac{9}{13}\begin{bmatrix} 6 \\ 3 \end{bmatrix}\right)$$

- Can E be written as a convex combination of A, B and D? If so, construct it.

  SOLUTION: No. The point E is above the convex hull of A, B and D (which is the triangle whose vertices are at A, B, D).

- Can A be written as a *linear* combination of A, B and D? If so, construct it.
  SOLUTION: Obvious typo there- I meant to say E can be written ...
  Using E, we can set up the matrix and solve:

$$\left[\begin{array}{ccc|c} 1 & 1 & 4 & 4 \\ 1 & 4 & 3 & 2 \end{array}\right] \quad \Rightarrow$$

  If the coefficients for the linear combination are c_1, c_2, c_3, we find them to be:

$$\begin{array}{lcl} C_1 & = & \frac{14}{3} - \frac{13}{3}C_3 \\ C_2 & = & -\frac{2}{3} + \frac{1}{3}C_3 \\ C_3 & = & \quad C_3 \end{array}$$

  Therefore, there are an infinite number of ways to make this linear combination (which was expected, since three vectors in $\mathbb{R}^2$ are not linearly independent).

10. Finish the definition: Two basic feasible solutions are said to be **adjacent** if:

    SOLUTION: Two basic feasible solutions are adjacent if they share all but one basic variable.

11. Let $\mathbf{d}$ be a direction of unboundedness. Using the *definition*, prove that this means that $r\mathbf{d}$ is also a direction of unboundedness, for any constant $r \geq 0$.

    SOLUTION: We assume an LP in standard form, so our set $S = \{\mathbf{x} | A\mathbf{x} = \mathbf{b}\}$. Then, $\mathbf{d} \neq 0$ is a direction of unboundedness for S if $\mathbf{x} + \lambda\mathbf{d} \in S$ for all $\mathbf{x} \in S$ and $\lambda \geq 0$.

    Therefore, in what is given, we can let $\mathbf{u} = r\mathbf{d}$ and show that $\mathbf{u}$ is a direction of unboundedness:

    Let $\mathbf{x}$ be any point of S and $\lambda \geq 0$. Then:

$$\mathbf{x} + \lambda\mathbf{u} = \mathbf{x} + (\lambda r)\,\mathbf{d}$$

    which must be in S since $\mathbf{d}$ was a direction of unboundedness.

12. If C is a convex set, then $\mathbf{d} \neq 0$ is a direction of unboundedness for C iff $\mathbf{x} + d \in C$ for all $\mathbf{x} \in C$ (Use the *definition* of unboundedness).

    SOLUTION: We have two directions-

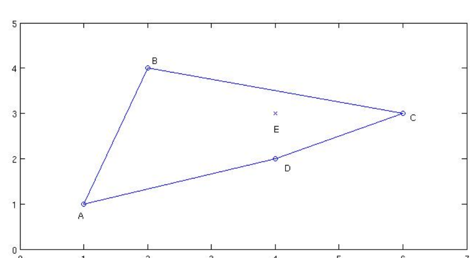


Figure 2: Figure for the convex combinations, Exercise 9.

- $\mathbf{d} \neq 0$ is a direction of unboundedness for C implies $\mathbf{x} + d \in C$ is trivially true, since we can just make $\lambda = 1$.
- We now show that, if $\mathbf{x}+\mathbf{d} \in C$ for all $\mathbf{x} \in C$, then $\mathbf{d}$ is a direction of unboundedness for C:

  Let $\mathbf{x}_0$ be any point in S, and $\lambda \geq 0$. Then show that $\mathbf{x}_0 + \lambda\mathbf{d} \in S$.

  Let $\lambda = N + \alpha$, where N is a non-negative integer, and $0 \leq \alpha < 1$. Then

  $$\mathbf{x}_0 + \lambda\mathbf{d} = \mathbf{x}_0 + (N+\alpha)\mathbf{d} = (\mathbf{x}_0 + \mathbf{d}) + ((N-1)+\alpha)\mathbf{d}$$

  Now, since $\mathbf{x}_1 = \mathbf{x}_0 + \mathbf{d}$, then $\mathbf{x}_1 \in C$, we can write this as:

  $$\mathbf{x}_1 + \mathbf{d} + ((N-2)+\alpha)\mathbf{d}$$

  and so on. Therefore, we have that $\mathbf{x}_0 + N\mathbf{d} \in S$ Finally, since $\mathbf{x}_0 + N\mathbf{d} + \mathbf{d} \in C$, then because C is convex, so will the vector $\mathbf{x}_0 + N\mathbf{d} + \alpha\mathbf{d} \in C$.

13. For an LP in standard form (see above), prove that the vector $\mathbf{d}$ is a direction of unboundedness iff $A\mathbf{d} = 0$ and $\mathbf{d} \geq 0$.

    Solution:

    - Show that if $A\mathbf{d} = \mathbf{0}$, with $\mathbf{d} \geq 0$, then $\mathbf{d}$ is a direction of unboundedness.

      Note that this means we have to show that $\mathbf{y} = \mathbf{x} + \lambda\mathbf{d} \in S$ for every λ. Let $\mathbf{x}$ be in the feasible set, $\mathbf{x} \in S$ so that $\mathbf{x} \geq 0$. Now,

      $$A\mathbf{y} = A(\mathbf{x} + \lambda\mathbf{d}) = A\mathbf{x} + \lambda A\mathbf{d} = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

      Wait! We're not done- Check that $\mathbf{y} \geq 0$ (it is since $\lambda, x, d \geq 0$).
    - Now go in the reverse: Suppose that we know that $\mathbf{d}$ is a direction of unboundedness. We show that $A\mathbf{d} = \mathbf{0}$ and $\mathbf{d} \geq 0$.

      Let $\mathbf{x}$ be in the feasible set. One path we could take is to suppose that, by way of contradiction, that $A\mathbf{d} \neq \mathbf{0}$. Then

      $$A(\mathbf{x} + \lambda\mathbf{d}) = A\mathbf{x} + \lambda A\mathbf{d} = \mathbf{b} + \lambda\mathbf{k} \neq \mathbf{b}$$

      But then $\mathbf{x} + \lambda\mathbf{d}$ is not an element of S (contradiction).

      The other part: Is $\mathbf{d} \geq 0$? If not, then at least one coordinate $d_i < 0$. But then it is possible to find λ so that the i^{th} coordinate of $\mathbf{x} + \lambda\mathbf{d}$ is negative (contradiction).

14. Show that the set of optimal solutions to an LP (assume in standard form) is convex.

    SOLUTION: Define $S = \left\{\mathbf{x} | A\mathbf{x} = \mathbf{b},\ \mathbf{c}^T\mathbf{x} = L\right\}$ Now, let $\mathbf{y}_1, \mathbf{y}_2 \in S$. We show that all points on the line segment between them is also in S. Let $\mathbf{y}$ be a point between- Then there is a $0 \leq t \leq 1$ so that:

    $$\mathbf{y} = t\mathbf{y}_1 + (1-t)\mathbf{y}_2$$

    Now, $\mathbf{y}$ is also feasible, since

    $$A\mathbf{y} = tA\mathbf{y}_1 + (1-t)A\mathbf{y}_2 = t\mathbf{b} + (1-t)\mathbf{b} = \mathbf{b}$$

    And $\mathbf{y}$ will give the same optimal value,

    $$\mathbf{c}^T\mathbf{y} = t\mathbf{c}^T\mathbf{y}_1 + (1-t)\mathbf{c}^T\mathbf{y}_2 = tL + (1-t)L = L$$

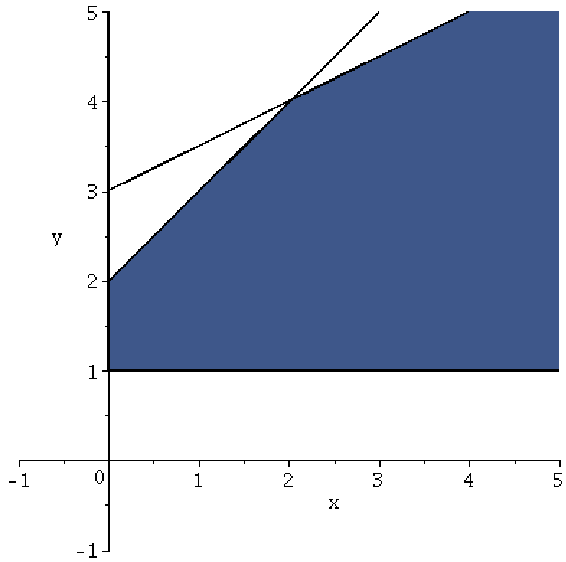
15. Let a feasible region be defined by the system of inequalities below:

$$\begin{aligned} -x_1 + 2x_2 &\leq 6 \\ -x_1 + x_2 &\leq 2 \\ x_2 &\geq 1 \\ x_1, x_2 \geq 0 & \end{aligned}$$

The point $(4, 3)$ is in the feasible region. Find vectors $\mathbf{d}$ and $\mathbf{b}_1, \cdots \mathbf{b}_k$ and constants σ_i so that the Representation Theorem is satisfied (NOTE: Your vector $\mathbf{x}$ from that theorem is more than two dimensional).

SOLUTION: Graphing the region in 2-d, we see that the extreme points are:

$$\mathbf{b}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$



And $\mathbf{d}$ can be any vector pointing outwards with a slope between 0 and 1/2. The easiest method to get the representation is to "aim backwards" at an extreme point, but using a vector that will be an allowable $\mathbf{d}$. In this case, we can write:

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

where the vector $(4, 1)$ has slope 1/4, so it is allowable. To write our system in 5 dimensions, we go back to the standard form:

$$\begin{aligned} -x_1 + 2x_2 + s_1 &= 6 \\ -x_1 + x_2 + s_2 &= 2 \\ x_2 - e_1 &= 1 \end{aligned}$$

so that, given x_1, x_2, we can compute the other variables- for vectors that are feasible. For vectors that are in the direction of unboundedness, remember to rewrite the equations replacing $(6, 2, 1)$ by $(0, 0, 0)$. Therefore,

$$\begin{bmatrix} 4 \\ 3 \\ 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

16. Let a feasible region be defined by the system of inequalities below:

$$\begin{aligned} -x_1 + x_2 &\leq 2 \\ x_1 - x_2 &\leq 1 \\ x_1 + x_2 &\leq 5 \\ x_1, x_2 \geq 0 & \end{aligned}$$

The point $(2,2)$ is in the feasible region. Find vectors $\mathbf{d}$ and $\mathbf{b}_1, \cdots \mathbf{b}_k$ and constants σ_i so that the Representation Theorem is satisfied (NOTE: Your vector $\mathbf{x}$ from that theorem is more than two dimensional).

SOLUTION: The point given is between two extreme points, $[0,2]^T$ and $[3,2]^T$. Therefore, in two dimensions we have

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = t \begin{bmatrix} 0 \\ 2 \end{bmatrix} + (1-t) \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \Rightarrow \quad t = \frac{1}{3}$$

We also get the matrix A in standard form, with column variables (in order): x_1, x_2, s_1, s_2, s_3, and

$$A = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{ll} s_1 & = x_1 - x_2 + 2 \\ s_2 & = -x_1 + x_2 + 1 \\ s_3 & = -x_1 - x_2 + 5 \end{array}$$

From which we get:

$$\begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 0 \\ 2 \\ 0 \\ 3 \\ 3 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} 3 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

17. The following two proofs go together:

(a) Suppose that $\lambda_1 \leq \lambda_2 \leq \cdots \lambda_n$, and let $\sigma_1, \cdots, \sigma_n$ be non-negative constants so that $\sum_{i=1}^{n} \sigma_i = 1$. Show that

$$\lambda_1 \leq \sigma_1 \lambda_1 + \sigma_2 \lambda_2 + \cdots \sigma_n \lambda_n \leq \lambda_n$$

SOLUTION:

$$\sigma_1 \lambda_1 + \sigma_2 \lambda_2 + \cdots \sigma_n \lambda_n \leq \sigma_1 \lambda_n + \sigma_2 \lambda_n + \cdots \sigma_n \lambda_n = (\lambda_n) \sum_i \sigma_i = \lambda_n$$

(A similar proof works the other way, too)

(b) Show that, if $\mathbf{x}$ is in the convex hull of vectors $\mathbf{b}_1, \cdots \mathbf{b}_k$, then for any constant vector $\mathbf{c}$,

$$\mathbf{c}^T \mathbf{x} \leq \max_i \left\{ \mathbf{c}^T \mathbf{b}_i \right\}$$

SOLUTION: If $\mathbf{x}$ is in the convex hull of the vectors $\mathbf{b}_i$, then we can write $\mathbf{x}$ as a convex combination of them:

$$\mathbf{x} = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_k \mathbf{b}_k$$

Taking the dot product of both sides with $\mathbf{c}$, we have:

$$\mathbf{c}^T \mathbf{x} = \alpha_1 (\mathbf{c}^T \mathbf{b}_1) + \cdots \alpha_k (\mathbf{c}^T \mathbf{b}_k)$$

which is the set up to the previous problem with $\lambda_j = \mathbf{c}^T\mathbf{b}_j$. By that exercise, we know that

$$\mathbf{c}^T\mathbf{x} = \alpha_1(\mathbf{c}^T\mathbf{b}_1) + \cdots \alpha_k(\mathbf{c}^T\mathbf{b}_k) \le \max_i \left\{\mathbf{c}^T\mathbf{b}_i\right\}$$

SIDE REMARK: Notice that this is a key element of the fundamental theorem of linear programming- The maximum and minimum of the objective function are attained at extreme points.

18. True or False, and explain: The Simplex Method will always choose a basic feasible solution that is **adjacent** to the current BFS.

    SOLUTION: That is true. It is because we will only replace one of the current basic variables with a new variable, therefore, the new BFS will keep all but one of the current set of basic variables.

19. Given the current tableau (with variables labeled above the respective columns), answer the questions below.

x_1	x_2	s_1	s_2	rhs
0	-1	0	2	24
0	$1/3$	1	$-1/3$	1
1	$2/3$	0	$1/3$	4

(a) Is the tableau optimal (and did your answer depend on whether we are maximizing or minimizing)? For the remaining questions, you may assume we are maximizing.

ANSWER: This tableau is not optimal for either. If we were minimizing, we could still pivot using s_2. If we were maximizing, we could still pivot in x_2.

(b) Give the current BFS.

ANSWER: The current BFS is $x_1 = 4, x_2 = 0, s_1 = 1$ and $s_2 = 0$.

(c) Directly from the tableau, can I increase x_2 from 0 to 1 and remain feasible? Can I increase it to 4?

ANSWER: From the ratio test, x_2 can be increased to 3 in the first, and 6 in the second. However, increasing it to 4 would violate the first constraint. Summary: I can increase x_2 from 0 to 1, but not to 4.

(d) If x_2 is increased from 0 to 1, compute the new value of z, x_1, s_1 (assuming s_2 stays zero).

SOLUTION:

$$z = 25 \qquad x_1 = \frac{10}{3} \qquad s_1 = \frac{2}{3}$$

(e) Write the objective function and all variables in terms of the non-basic (or free) variables, and then put them in vector form.

SOLUTION: For the current tableau, $z = 24 + x_2 - 2s_2$, with

$$\begin{array}{ll} x_1 & = 4 - 2/3x_2 - 1/3s_2 \\ x_2 & = \qquad x_2 \\ s_1 & = 1 - 1/3x_2 + 1/3s_2 \\ s_2 & = \qquad\qquad s_2 \end{array} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \frac{x_2}{3}\begin{bmatrix} -2 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \frac{s_2}{3}\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

20. Solve first using big-M, then repeat using the two-phase method.

$$\begin{array}{rrrl} \max\ z = & 5x_1 & -x_2 & \\ \text{st} & 2x_1 & +x_2 & = 6 \\ & x_1 & +x_2 & \leq 4 \\ & & x_2 & \leq 3 \\ & x_1, & x_2 & \geq 0 \end{array}$$

SOLUTION: This is Exercise 3 from the Chapter 4 Review. Below we'll list the initial tableau, the end of Phase I, and the final tableau. There will be two Row 0's (the top one for the Big-M, the second one for two-phase):

x_1	x_2	s_1	s_2	a_1	rhs
-5	1	0	0	M	0
0	0	0	0	1	0
2	1	0	0	1	6
1	1	1	0	0	4
1	2	0	1	0	5

$\Rightarrow$

x_1	x_2	s_1	s_2	a_1	rhs
0	7/2	0	0	$5/2+M$	15
0	0	0	0	1	0
1	1/2	0	0	1/2	3
0	1/2	1	0	$-1/2$	1
0	3/2	0	1	$-1/2$	2

On the left, we re-write the Row 0 to start Phase Two, and the left is what we get after making x_1 basic again.

x_1	x_2	s_1	s_2	a_1	rhs
0	7/2	0	0	$5/2+M$	15
-5	1	0	0	1	0
1	1/2	0	0	1/2	3
0	1/2	1	0	$-1/2$	1
0	3/2	0	1	$-1/2$	2

$\Rightarrow$

x_1	x_2	s_1	s_2	a_1	rhs
0	7/2	0	0	$5/2+M$	15
0	7/2	0	0	7/2	15
1	1/2	0	0	1/2	3
0	1/2	1	0	$-1/2$	1
0	3/2	0	1	$-1/2$	2

In both cases, we have a unique optimal solution: $x_1 = 3, x_2 = 0$, with $z = 15$.

21. Using the big-M method on a maximization problem, I got the following tableau:

	x_1	x_2	x_3	s_1	e_1	e_2	a_1	a_2	rhs
	$-1/2+2M$	$-5/2+M$	M	$1/2+M$	M	M	0	0	$2-3M$
x_3	1/2	1/2	1	1/2	0	0	0	0	2
a_1	$-3/2$	$-1/2$	0	$-1/2$	-1	0	1	0	2
a_2	$-1/2$	$-1/2$	0	$-1/2$	0	-1	0	1	1

Should I stop or should I go? If I stop, what should I conclude?

SOLUTION: Stop- Remember that M is a very large positive number, so there is nowhere left to pivot. Because the artificial variable is still a basic variable, that means the original feasible set is empty (no solution to the original problem).

22. Here's a tableau that we've obtained from using the Simplex Method. Answer the questions below about it.

x_1	x_2	x_3	s_1	s_2	s_3	rhs
1	0	0	0	3	-2	50
4	0	0	1	-1	0	5
1	1	0	0	1	-2	10
0	0	1	0	1	-1	15

(a) Is this tableau terminal (has the Simplex Method stopped)? If so, interpret the solution shown. If not, continue until you stop.

SOLUTION: Yes, this is a terminal solution. The LP is unbounded.

(b) Write down the system of equations that this tableau represents (be sure to write BVs in terms of NBVs).

SOLUTION: The basic variables are x_2, x_3 and s_1.

$$\begin{array}{lllll} x_1 & = & x_1 & & \\ x_2 & = 10 & -x_1 & -s_2 & +2s_3 \\ x_3 & = 15 & & -s_2 & +s_3 \\ s_1 & = 5 & -4x_1 & +s_2 & \\ s_2 & = & & s_2 & \\ s_3 & = & & & s_3 \end{array}$$

The objective function is given by: $z = 50 - x_1 - 3s_2 + 2s_3$.

(c) Given the tableau shown, the current basic variables are s_1, x_2, x_3. Is it possible that the **previous** set of basic variables were: s_1, s_2, x_3? To see, compute the previous Row 0. (Hint: You want to replace or substitute x_2 with s_2 as basic).

SOLUTION: In replacing x_2 with s_2, we can use the second equation, and switch them:

$$s_2 = 10 - x_1 - x_2 + 2s_3$$

The new objective function would then be:

$$z = 50 - x_1 - 3(10 - x_1 - x_2 + 2s_3) + 2s_3 = 20 + 2x_1 + 3x_2 - 4s_3$$

Therefore, the previous Row 0 would have been: $[-2\ -3\ 0\ 0\ 0\ -4]$, so yes, that is a possibility.

Chapter 3 Review, #18

We'll need to keep track of both the district and the school, which suggests a double index. For the two schools, we'll take Cooley as 1 and Whitman as 2.

Now we can define two sets of variables- one for minority students and one for other students- Respectively, we'll use M and N so that

$$M_{ij} = \text{ Number of minority students who live in district } i \text{ and attend school } j$$

So that the other students will be denoted by N_{ij}.

The miles traveled using Table 54 will give the following, and the objective function will be the sum of the six values:

District	School 1	School 2
1	$1 \cdot (M_{11} + N_{11})$	$2 \cdot (M_{12} + N_{12})$
2	$2 \cdot (M_{21} + N_{21})$	$1 \cdot (M_{22} + N_{22})$
3	$1 \cdot (M_{31} + N_{31})$	$1 \cdot (M_{32} + N_{32})$

The purpose of Table 53 is to put some values on the variables- These will be equalities since the number of students in each case is known. From the table, we have:

District	Minority Students	Nonminority Students
1	$M_{11} + M_{12} = 50$	$N_{11} + N_{12} = 200$
2	$M_{21} + M_{22} = 50$	$N_{21} + N_{22} = 250$
3	$M_{31} + M_{32} = 100$	$N_{31} + N_{32} = 150$

The stuff written about percentages means that the percentage of minority students in each school should be between 20 and 30 percent. For Cooley High, that means:

$$0.20 \le \frac{M_{11} + M_{21} + M_{31}}{M_{11} + M_{21} + M_{31} + N_{11} + N_{21} + N_{31}} \le 0.30$$

Similarly,

$$0.20 \le \frac{M_{12} + M_{22} + M_{32}}{M_{12} + M_{22} + M_{32} + N_{12} + N_{22} + N_{32}} \le 0.30$$

And we should have between 300 and 500 students at each school:

$$300 \le M_{11} + N_{11} + M_{21} + N_{21} + M_{31} + N_{31} \le 500$$

$$300 \le M_{12} + N_{12} + M_{22} + N_{22} + M_{32} + N_{32} \le 500$$

Note that each double inequality should be written out as two inequalities, the fractions ought to be simplified to linear constraints, and all variables are non-negative.