Homework, Section 6.10 (Complementary Slackness)

1. Consider the LP:

$$\begin{aligned} \max z = \quad & 2x_1 + 5x_2 + 3x_3 \\ \text{st} \quad & 2x_1 + x_2 + 2x_3 \leq 10 \\ & (3/2)x_1 + 6x_2 - 2x_3 \leq 18 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

   (a) State the dual.
   SOLUTION:

$$\begin{aligned} \min w = \quad & 10y_1 + 18y_2 \\ \text{st} \quad & 2y_1 + (3/2)y_2 \geq 2 \\ & y_1 + 6y_2 \geq 5 \\ & 2y_1 - 2y_2 \geq 3 \\ & y_1, y_2 \geq 0 \end{aligned}$$

   (b) Given that $x_1 = 0, x_2 = 3, x_3 = 0$ is an basic solution to the LP, use complementary slackness to find the complementary basic solution to the dual.
   SOLUTION: Looking at our complementary variables (and solving for slack s_1, s_2), we have

x_1	x_2	x_3	s_1	s_2
0	3	0	7	0
e_1	e_2	e_3	y_1	y_2
	0		0	

   This means that the second constraint of the dual is actually an equality. With $y_1 = 0$, we see that $y_2 = 5/6$. We can then finish the excess variables to fill in the entire picture:

x_1	x_2	x_3	s_1	s_2
0	3	0	7	0
e_1	e_2	e_3	y_1	y_2
$-3/4$	0	$-14/3$	0	$5/6$

   (And we note that this is not feasible for the dual).

   (c) Given that $x_1 = 4, x_2 = 2, x_3 = 0$ is an basic solution to the LP, use complementary slackness to find the complementary basic solution to the dual. Are these solutions optimal?
   SOLUTION: Looking at our complementary variables (and solving for slack s_1, s_2), we have

x_1	x_2	x_3	s_1	s_2
4	2	0	0	0
e_1	e_2	e_3	y_1	y_2
0	0			

This means that the first two inequalities of the dual are equalities, so then we can solve the corresponding system for y_1, y_2.

$$\begin{aligned} 2y_1 + \tfrac{3}{2}y_2 &= 2 \\ y_1 + 6y_2 &= 5 \end{aligned} \quad \Rightarrow \quad \begin{aligned} y_1 &= 3/7 \\ y_2 &= 16/21 \end{aligned}$$

From those, we see $e_3 = -2/3$, so this is not a feasible solution for the dual.

(d) Given that $x_1 = 0, x_2 = 4, x_3 = 3$ is an basic solution to the LP, use complementary slackness to find the complementary basic solution to the dual. Are these solutions optimal?

SOLUTION: From what is given, we can inspect our complementary variables and by complementary slackness, we can insert some zeros:

x_1	x_2	x_3	s_1	s_2
0	4	3	0	0
e_1	e_2	e_3	y_1	y_2
	0	0		

This says that the second and third constraints for the dual are equalities, from which we can solve for y_1, y_2, and we get $y_1 = 2$ and $y_2 = 1/2$, and from those we can compute e_3 to finish our table:

x_1	x_2	x_3	s_1	s_2
0	4	3	0	0
e_1	e_2	e_3	y_1	y_2
11/4	0	0	2	1/2

This is optimal, since we have a basic solution that is feasible for both the primal and the dual. Checking, we see that $z = w = 29$.

2. Consider the LP:

$$\begin{aligned} \max z = \quad & 5x_1 + 10x_2 \\ \text{st} \quad & x_1 + 3x_2 \leq 50 \\ & 4x_1 + 2x_2 \leq 60 \\ & x_1 \leq 5 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(a) State the dual.

SOLUTION: The dual is given by:

$$\begin{aligned} \min w = \quad & 50y_1 + 60y_2 + 5y_3 \\ \text{st} \quad & y_1 + 4y_2 + y_3 \geq 5 \\ & 3y_1 + 2y_2 \geq 10 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

(b) Given that $x_1 = 5, x_2 = 15$ is an optimal solution to the LP, use complementary slackness to find the optimal solution to the dual.

SOLUTION: Same reasoning as the last question- first look at the complementary variables and locate the zeros.

x_1	x_2	s_1	s_2	s_3
5	15	0	10	0
e_1	e_2	y_1	y_2	y_3
0	0		0	

From this, we see that the two inequalities for the dual must be equalities. Setting $y_2 = 0$, we can solve for y_1, y_3. That gives

x_1	x_2	s_1	s_2	s_3
5	15	0	10	0
e_1	e_2	y_1	y_2	y_3
0	0	10/3	0	5/3

These should be optimal since they're both feasible. Verify that $z = w$, and we see that this is the case- $z = w = 175$.