

# Example

$$\begin{array}{ll} \max & 6x_1 + 5x_2 \\ \text{s.t.} & x_1 + x_2 \leq 5 \\ & 3x_1 + 2x_2 \leq 12 \end{array}$$

with  $x_1, x_2 \geq 0$ . Find an upper bound to the maximum.

- Using constraint 2 (multiply by 3):

$$6x_1 + 5x_2 \leq 9x_1 + 6x_2 \leq 36$$

- Using constraint 1 (multiply by 6):

$$6x_1 + 5x_2 \leq 6x_1 + 6x_2 \leq 30$$

## Example

$$\begin{array}{ll}
 \max & 6x_1 + 5x_2 \\
 \text{s.t.} & x_1 + x_2 \leq 5 \\
 & 3x_1 + 2x_2 \leq 12
 \end{array}$$

How about a linear combination?

$$\begin{array}{r}
 y_1(x_1 \quad + x_2) \leq 5y_1 \\
 + y_2(3x_1 \quad + 2x_2) \leq 12y_2 \\
 \hline
 (y_1 + 3y_2)x_1 + (y_1 + 2y_2)x_2 \leq 5y_1 + 12y_2
 \end{array}$$

What needs to be true for this to give an upper bound? (The Dual)

$$\begin{array}{ll}
 \min & 5y_1 + 12y_2 \\
 \text{s.t.} & y_1 + 3y_2 \geq 6 \\
 & y_1 + 2y_2 \geq 5
 \end{array}$$

with  $y_1, y_2 \geq 0$ .

We showed that the primal, dual are:

$$\begin{array}{ll}
 \max & 6x_1 + 5x_2 \\
 \text{s.t.} & x_1 + x_2 \leq 5 \\
 & 3x_1 + 2x_2 \leq 12
 \end{array}
 \iff
 \begin{array}{ll}
 \min & 5y_1 + 12y_2 \\
 \text{s.t.} & y_1 + 3y_2 \geq 6 \\
 & y_1 + 2y_2 \geq 5
 \end{array}$$

Generalized, we have a “normal” max problem:

$$\begin{array}{ll}
 \max z = & \mathbf{c}^T \mathbf{x} \\
 \text{st} & \mathbf{Ax} \leq \mathbf{b} \\
 & \mathbf{x} \geq 0
 \end{array}
 \iff
 \begin{array}{ll}
 \min w = & \mathbf{b}^T \mathbf{y} \\
 \text{st} & \mathbf{A}^T \mathbf{y} \geq \mathbf{c} \\
 & \mathbf{y} \geq 0
 \end{array}$$

Note:  $\mathbf{b}$  may have negative values in this construction.

$$\begin{array}{ll}
 \max z = & \mathbf{c}^T \mathbf{x} \\
 \text{st} & A\mathbf{x} \leq \mathbf{b} \\
 & \mathbf{x} \geq 0
 \end{array}
 \iff
 \begin{array}{ll}
 \min w = & \mathbf{b}^T \mathbf{y} \\
 \text{st} & A^T \mathbf{y} \geq \mathbf{c} \\
 & \mathbf{y} \geq 0
 \end{array}$$

How many constraints are in the dual?

*Same as number of variables in primal.*

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*Same as number of constraints in primal.*

- Every primal has a dual.
- A maximum becomes a minimum (dual of dual is primal).
- Variable in primal  $\rightarrow$  constraint in dual.
- Constraint in primal  $\rightarrow$  variable in dual.
- The RHS of primal  $\rightarrow$  obj function coeffs in dual.

## Self test: Create the Dual

$$\begin{array}{llll} \max & 2x_1 & +3x_2 & -x_3 \\ \text{s.t.} & x_1 & +2x_2 & +x_3 \leq 1 \\ & x_1 & -x_2 & -x_3 \leq 5 \end{array}$$

 $\iff$ 

$$\begin{array}{ll} \min & y_1 + 5y_2 \\ \text{s.t.} & y_1 + y_2 \geq 2 \\ & 2y_1 - y_2 \geq 3 \\ & y_1 - y_2 \geq -1 \end{array}$$

An example with non-“normal” issues. Find the dual for:

$$\begin{aligned} \min \quad & 8x_1 + 5x_2 + 4x_3 \\ \text{s.t.} \quad & 4x_1 + 2x_2 + 8x_3 = 12 \\ & 7x_1 + 5x_2 + 6x_3 \geq 9 \\ & 8x_1 + 5x_2 + 4x_3 \leq 10 \\ & 3x_1 + 7x_2 + 9x_3 \geq 7 \\ & x_1 \geq 0, x_2 \text{ URS}, x_3 \leq 0 \end{aligned}$$

We'll put this system into “normal form”:

- Change the min to a max.
- Multiply Constraints 2 and 4 by  $-1$ .
- Change  $=$  into inequalities.
- Find substitutions for  $x_2, x_3$ .

## First three issues:

$$\begin{array}{rcll}
 \text{max} & & -8x_1 & -5x_2 & -4x_3 \\
 \\ 
 \text{s.t.} & & 4x_1 & +2x_2 & +8x_3 & \leq 12 \\
 & & -4x_1 & -2x_2 & -8x_3 & \leq -12 \\
 & & -7x_1 & -5x_2 & -6x_3 & \leq -9 \\
 & & 8x_1 & +5x_2 & +4x_3 & \leq 10 \\
 & & -3x_1 & -7x_2 & -9x_3 & \leq -7
 \end{array}$$

$$x_1 \geq 0, x_2 \text{ URS}, x_3 \leq 0$$

Last issue: Let  $x_2 = x_4 - x_5$  and  $x_3 = -x_6$  (with all these new vars non-neg)

Now a summary of the equations using a table:

$x_1 \geq 0$	$x_4 \geq 0$	$x_5 \geq 0$	$x_6 \geq 0$	
4	2	-2	-8	$\leq 12$
-4	-2	2	8	$\leq -12$
-7	-5	5	6	$\leq -9$
8	5	-5	-4	$\leq -12$
-3	-7	7	9	$\leq -7$
$\geq -8$	$\geq -5$	$\geq 5$	$\geq 4$	

Let  $p_1, \dots, p_5$  be the new vars:

$$\begin{aligned}
 \min w = & 12p_1 - 12p_2 - 9p_3 + 10p_4 - 7p_5 \\
 & 4p_1 - 4p_2 - 7p_3 + 8p_4 - 3p_5 \geq -8 \\
 & 2p_1 - 2p_2 - 5p_3 + 5p_4 - 7p_5 \geq -5 \\
 & -2p_1 + 2p_2 + 5p_3 - 5p_4 + 7p_5 \geq 5 \\
 & -8p_1 + 8p_2 + 6p_3 - 4p_4 + 9p_5 \geq 4 \\
 & p_1, p_2, p_3, p_4, p_5 \geq 0
 \end{aligned}$$

$$\begin{aligned}\min w = & 12(p_1 - p_2) - 9p_3 + 10p_4 - 7p_5 \\ & 4(p_1 - p_2) - 7p_3 + 8p_4 - 3p_5 \geq -8 \\ & 2(p_1 - p_2) - 5p_3 + 5p_4 - 7p_5 \geq -5 \\ & -2(p_1 - p_2) + 5p_3 - 5p_4 + 7p_5 \geq 5 \\ & -8(p_1 - p_2) + 6p_3 - 4p_4 + 9p_5 \geq 4 \\ & p_1, p_2, p_3, p_4, p_5 \geq 0\end{aligned}$$

- Variables  $p_1, p_2$  can be combined:  $p_6 = p_1 - p_2$  (now is URS).
- Constraints 2 and 3 can be combined into equality.
- Multiply first constraint by  $-1$  to get positive  $b$ .

$$\begin{array}{rcllcl}
 \min w = & 12p_6 & -9p_3 & +10p_4 & -7p_5 & \\
 & -4p_6 & +7p_3 & -8p_4 & +3p_5 & \leq 8 \\
 & -2p_6 & +5p_3 & -5p_4 & +7p_5 & = 5 \\
 & -8p_6 & +6p_3 & -4p_4 & +9p_5 & \geq 4
 \end{array}$$

with  $p_6$  URS, and  $p_3, p_4, p_5 \geq 0$ .

- Let  $p_7 = -p_6$  and  $p_8 = -p_4$
- $p_7$  is URS,  $p_8 \leq 0$ ,  $p_3, p_5 \geq 0$ .

## Changing notation slightly:

$$\max z = 12x_1 + 9x_2 + 10x_3 + 7x_4$$

$$\text{s.t. } 4x_1 + 7x_2 + 8x_3 + 3x_4 \leq 8$$

$$2x_1 + 5x_2 + 5x_3 + 7x_4 = 5$$

$$8x_1 + 6x_2 + 4x_3 + 9x_4 \geq 4$$

$$x_1 \text{ URS, } x_2 \geq 0, x_3 \leq 0, x_4 \geq 0$$

$$\min w = 8y_1 + 5y_2 + 4y_3$$

$$\text{s.t. } 4y_1 + 2y_2 + 8y_3 = 12$$

$$7y_1 + 5y_2 + 6y_3 \geq 9$$

$$8y_1 + 5y_2 + 4y_3 \leq 10$$

$$3y_1 + 7y_2 + 9y_3 \geq 7$$

$$y_1 \geq 0, y_2 \text{ URS, } y_3 \leq 0$$

## Summary:

Primal:	Dual:	
max	min	normal
$\leq$ constraint	$\geq 0$ variable	normal
$\geq$ constraint	$\leq 0$ variable	
Equality constraint	URS variable	
$\geq 0$ variable	$\geq$ constraint	normal
$\leq 0$ variable	$\leq$ constraint	
URS variable	Equality constraint	

# Primal-Dual Table

$$\begin{array}{ll}
 \max z = \mathbf{c}^T \mathbf{x} & \\
 \text{st } A\mathbf{x} \leq \mathbf{b} & \iff \\
 \mathbf{x} \geq 0 & 
 \end{array}
 \qquad
 \begin{array}{ll}
 \min w = \mathbf{b}^T \mathbf{y} & \\
 \text{st } A^T \mathbf{y} \geq \mathbf{c} & \\
 \mathbf{y} \geq 0 & 
 \end{array}$$

Main concerns:

If the inequalities in the primal are not normal, what happens in the dual?

If the variables are not normal in the primal, what happens in the dual?

(and vice-versa)

	$x_1?$	$x_2?$	$\dots$	$x_n?$	
$y_1?$	$A$				$b_1?$
$y_2?$					$b_2?$
$\vdots$					$\vdots$
$y_m?$					$b_m?$
	$c_1?$	$c_2?$	$\dots$	$c_n?$	

## Using a table

Starting problem:

$$\begin{aligned}
 \min w &= 8y_1 + 5y_2 + 4y_3 \\
 \text{s.t. } &4y_1 + 2y_2 + 8y_3 = 12 \\
 &7y_1 + 5y_2 + 6y_3 \geq 9 \quad \text{with } y_1 \geq 0, y_2 \text{ URS}, y_3 \leq 0 \\
 &8y_1 + 5y_2 + 4y_3 \leq 10 \\
 &3y_1 + 7y_2 + 9y_3 \geq 7
 \end{aligned}$$

(Asterisks mark things that are different than “normal”)

	$x_1?$	$x_2?$	$x_3?$	$x_4?$	
$y_1 \geq 0$	4	7	8	3	?8
$y_2 \text{ urs}(*)$	2	5	5	7	?5
$y_3 \leq 0(*)$	8	6	4	9	?4
	= 12(*)	$\geq 9$	$\leq 10(*)$	$\geq 7$	

## Using a table

	$x_1?$	$x_2?$	$x_3?$	$x_4?$	
$y_1 \geq 0$	4	7	8	3	$\leq 8$
$y_2$ urs(*)	2	5	5	7	?5
$y_3 \leq 0$ (*)	8	6	4	9	?4
	$= 12(*)$	$\geq 9$	$\leq 10(*)$	$\geq 7$	

## Using a table

	$x_1?$	$x_2?$	$x_3?$	$x_4?$	
$y_1 \geq 0$	4	7	8	3	$\leq 8$
$y_2$ urs(*)	2	5	5	7	$= 5$
$y_3 \leq 0$ (*)	8	6	4	9	?4
	$= 12$ (*)	$\geq 9$	$\leq 10$ (*)	$\geq 7$	

## Using a table

	$x_1?$	$x_2?$	$x_3?$	$x_4?$	
$y_1 \geq 0$	4	7	8	3	$\leq 8$
$y_2 \text{ urs}(*)$	2	5	5	7	$= 5$
$y_3 \leq 0(*)$	8	6	4	9	$\geq 4$
	$= 12(*)$	$\geq 9$	$\leq 10(*)$	$\geq 7$	

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$y_1 \geq 0$	4	7	8	3	$\leq 8$
$y_2$ urs(*)	2	5	5	7	$= 5$
$y_3 \leq 0$ (*)	8	6	4	9	$\geq 4$
	$= 12$ (*)	$\geq 9$	$\leq 10$ (*)	$\geq 7$	

## Using a table

	$x_1$ URS	$x_2?$	$x_3?$	$x_4?$	
$y_1 \geq 0$	4	7	8	3	$\leq 8$
$y_2 \text{ urs}(* )$	2	5	5	7	$= 5$
$y_3 \leq 0(* )$	8	6	4	9	$\geq 4$
	$= 12(* )$	$\geq 9$	$\leq 10(* )$	$\geq 7$	

## Using a table

	$x_1$ URS	$x_2?$	$x_3?$	$x_4?$	
$y_1 \geq 0$	4	7	8	3	$\leq 8$
$y_2 \text{ urs}(* )$	2	5	5	7	$= 5$
$y_3 \leq 0(* )$	8	6	4	9	$\geq 4$
	$= 12(* )$	$\geq 9$	$\leq 10(* )$	$\geq 7$	

## Using a table

	$x_1$ URS	$x_2 \geq 0$	$x_3?$	$x_4?$	
$y_1 \geq 0$	4	7	8	3	$\leq 8$
$y_2 \text{ urs}(* )$	2	5	5	7	$= 5$
$y_3 \leq 0(* )$	8	6	4	9	$\geq 4$
	$= 12(* )$	$\geq 9$	$\leq 10(* )$	$\geq 7$	

## Using a table

	$x_1$ URS	$x_2 \geq 0$	$x_3?$	$x_4?$	
$y_1 \geq 0$	4	7	8	3	$\leq 8$
$y_2$ urs(*)	2	5	5	7	$= 5$
$y_3 \leq 0(*)$	8	6	4	9	$\geq 4$
	$= 12(*)$	$\geq 9$	$\leq 10(*)$	$\geq 7$	

## Using a table

	$x_1$ URS	$x_2 \geq 0$	$x_3 \leq 0$	$x_4?$	
$y_1 \geq 0$	4	7	8	3	$\leq 8$
$y_2$ urs(*)	2	5	5	7	$= 5$
$y_3 \leq 0(*)$	8	6	4	9	$\geq 4$
	$= 12(*)$	$\geq 9$	$\leq 10(*)$	$\geq 7$	

## Using a table

	$x_1$ URS	$x_2 \geq 0$	$x_3 \leq 0$	$x_4?$	
$y_1 \geq 0$	4	7	8	3	$\leq 8$
$y_2$ urs(*)	2	5	5	7	$= 5$
$y_3 \leq 0(*)$	8	6	4	9	$\geq 4$
	$= 12(*)$	$\geq 9$	$\leq 10(*)$	$\geq 7$	

## Using a table

	$x_1$ URS	$x_2 \geq 0$	$x_3 \leq 0$	$x_4 \geq 0$	
$y_1 \geq 0$	4	7	8	3	$\leq 8$
$y_2$ urs(*)	2	5	5	7	$= 5$
$y_3 \leq 0$ (*)	8	6	4	9	$\geq 4$
	$= 12$ (*)	$\geq 9$	$\leq 10$ (*)	$\geq 7$	

## Using a table- Here are Primal and Dual

	$x_1$ URS	$x_2 \geq 0$	$x_3 \leq 0$	$x_4 \geq 0$	
$y_1 \geq 0$	4	7	8	3	$\leq 8$
$y_2$ urs(*)	2	5	5	7	$= 5$
$y_3 \leq 0(*)$	8	6	4	9	$\geq 4$
	$= 12(*)$	$\geq 9$	$\leq 10(*)$	$\geq 7$	

$$\max z = 12x_1 + 9x_2 + 10x_3 + 7x_4$$

$$\text{s.t. } 4x_1 + 7x_2 + 8x_3 + 3x_4 \leq 8$$

$$2x_1 + 5x_2 + 5x_3 + 7x_4 = 5$$

$$8x_1 + 6x_2 + 4x_3 + 9x_4 \geq 4$$

$$x_1 \text{ URS}, x_2 \geq 0, x_3 \leq 0, x_4 \geq 0$$

$$\min w = 8y_1 + 5y_2 + 4y_3$$

$$\text{s.t. } 4y_1 + 2y_2 + 8y_3 = 12$$

$$7y_1 + 5y_2 + 6y_3 \geq 9$$

$$8y_1 + 5y_2 + 4y_3 \leq 10$$

$$3y_1 + 7y_2 + 9y_3 \geq 7$$

$$y_1 \geq 0, y_2 \text{ URS}, y_3 \leq 0$$

## Example 2

Use a table to write the dual:

$$\begin{aligned}
 \max z = & 2x_1 + x_2 \\
 \text{st } & x_1 + x_2 = 2 \\
 & 2x_1 - x_2 \geq 3 \\
 & x_1 - x_2 \leq 1 \\
 & x_1 \geq 0, x_2 \text{ urs}
 \end{aligned}$$

	$x_1 \geq 0$	$x_2 \text{ urs} (*)$	
$y_1 ?$	1	1	$= 2(*)$
$y_2 ?$	2	-1	$\geq 3(*)$
$y_3 ?$	1	-1	$\leq 1$
	? 2	? 1	

	$x_1 \geq 0$	$x_2 \text{ urs} (*)$	
$y_1 \text{ urs}$	1	1	$= 2(*)$
$y_2 \leq 0$	2	-1	$\geq 3(*)$
$y_3 \geq 0$	1	-1	$\leq 1$
	$\geq 2$	$= 1$	

Use a table to write the dual:

$$\begin{aligned}
 \max z = & 2x_1 + x_2 \\
 \text{st} \quad & x_1 + x_2 = 2 \\
 & 2x_1 - x_2 \geq 3 \\
 & x_1 - x_2 \leq 1 \\
 & x_1 \geq 0, x_2 \text{ urs}
 \end{aligned}$$

	$x_1 \geq 0$	$x_2 \text{ urs} (*)$	
$y_1 \text{ urs}$	1	1	$= 2(*)$
$y_2 \leq 0$	2	-1	$\geq 3(*)$
$y_3 \geq 0$	1	-1	$\leq 1$
	$\geq 2$	$= 1$	

$$\begin{aligned}
 \min w = & 2y_1 + 3y_2 + y_3 \\
 \text{st} \quad & y_1 + 2y_2 + y_3 \geq 2 \\
 & y_1 - y_2 - y_3 = 1 \\
 & y_1 \text{ urs}, y_2 \leq 0, y_3 \geq 0
 \end{aligned}$$

## Last Example

Find the dual associated with the following primal:

$$\begin{array}{ll} \max & z = 3x_1 + x_2 \\ \text{st} & 2x_1 + x_2 \leq 4 \\ & 3x_1 + 2x_2 \geq 6 \\ & 4x_1 + 2x_2 = 7 \\ & x_1, x_2 \geq 0 \end{array}$$

## Solution:

	$x_1 \geq 0$	$x_2 \geq 0$	
$y_1 \geq 0$	2	1	$\leq 4$
$y_2 \leq 0$	3	2	$\geq 6(*)$
$y_3$ URS	4	2	$= 7(*)$
	$\geq 3$	$\geq 1$	

$$\begin{aligned}
 \max \quad & z = 4y_1 + 6y_2 + 7y_3 \\
 \text{st} \quad & 2y_1 + 3y_2 + 4y_3 \geq 3 \\
 & y_1 + 2y_2 + 2y_3 \geq 1 \\
 & y_1 \geq 0, y_2 \leq 0, y_3 \text{ URS}
 \end{aligned}$$