6.7: The Dual Theorem, Part 1

Before we start, we look at several Lemmas to get the ball rolling. To set up the notation, we will assume the normal form of our primal and dual,

$$\begin{array}{rrl} & \text{Primal:} & \\ \max & z = & \mathbf{c}^T\mathbf{x} \\ & \text{st} & A\mathbf{x} \le \mathbf{b} \\ & & \mathbf{x} \ge 0 \end{array} \qquad \begin{array}{rrl} & \text{Dual:} & \\ \min & w = & \mathbf{b}^T\mathbf{y} \\ & \text{st} & A^T\mathbf{y} \ge \mathbf{c} \\ & & \mathbf{y} \ge 0 \end{array}$$

where A is $m \times n$ with rank m.

1. RECALL: $(AB)^T = B^T A^T$ (where A, B could be vectors or arrays).
2. **Exercise:** Assume all vectors are in $\mathbb{R}^n$.

   If $\mathbf{x} \ge \mathbf{c}$, then is $\mathbf{y}^T\mathbf{x} \ge \mathbf{y}^T\mathbf{c}$? Give conditions for which it is true, and prove it.

   SOLUTION: It is true if $\mathbf{y} \ge 0$.
3. **Lemma 1:** (Weak Duality) If we choose any feasible point for the primal, $\mathbf{x}$, and any feasible point for the dual, $\mathbf{y}$, then $z \le w$.

   Proof: We note that this means we need to show $\mathbf{c}^T\mathbf{x} \le \mathbf{b}^T\mathbf{y}$.

   If $\mathbf{x}, \mathbf{y}$ are each feasible, then we know that

   $$A\mathbf{x} \le \mathbf{b} \quad \text{and} \quad A^T\mathbf{y} \le \mathbf{c}$$

   Now, $\mathbf{x} \ge 0$ and $\mathbf{y} \ge 0$, so we can dot the first with $\mathbf{y}$ and the second by $\mathbf{x}$:

   $$\mathbf{y}^T(A\mathbf{x}) \le \mathbf{y}^T\mathbf{b} \qquad \text{and} \qquad \mathbf{x}^T(A^T\mathbf{y}) \ge \mathbf{x}^T\mathbf{c}$$

   And with this:

   $$\mathbf{y}^T(A\mathbf{x}) = \mathbf{x}^T(A^T\mathbf{y})$$

   we get:

   $$\mathbf{x}^T\mathbf{c} \le \mathbf{y}^T A\mathbf{x} \le \mathbf{y}^T\mathbf{b}$$
4. **Lemma 2:** (Strong Duality) Let $\mathbf{x}$, $\mathbf{y}$ be any feasible points to the primal and dual, respectively, so that

   $$\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$$

   Then, the solutions are optimal for their respective LPs.

   Note: This lemma does not guarantee that such points exist.

   Proof: We know that, for any $\mathbf{y}$ that is feasible for the dual, and any $\mathbf{x}$ feasible for the primal, then $\mathbf{c}^T\mathbf{x} \le \mathbf{b}^T\mathbf{y}$.

   Now, consider the $\mathbf{y}$ for which $z = w$. Then this is an upper bound for all $\mathbf{c}^T\mathbf{x}$, so there is no $\mathbf{x}$ that is larger (so the given $\mathbf{x}$ is a maximizer).

   Similarly, $\mathbf{c}^T\mathbf{x}$ is a lower bound for all $\mathbf{y}$, so no $\mathbf{y}$ vector will yield a smaller value (so the given $\mathbf{y}$ is a minimizer).

5. Lemmas 3 and 4:

- If the primal is unbounded, then the dual is infeasible. (HW 7) (The contrapositive is easy: If there is any feasible point to the dual, then $\mathbf{b}^T\mathbf{y}$ would be an upper bound for any feasible point in the primal).
- If the dual is unbounded, then the primal is infeasible. (HW 8)
- It is possible that both the primal and dual are infeasible.