The Dual Theorem, Part II

The Dual Theorem

Let $\mathcal{B}$ be an optimal basis for the primal. Then

$$\mathbf{y} = (\mathbf{c}_{\mathcal{B}}^T B^{-1})^T$$

is an optimal solution to the dual. Furthermore, the optimal values of the primal and dual are equal ($z = w$).

Side Remark: We could add an extra sentence to this theorem: If both problems are feasible, then both problems have finite optimal solutions.

Proof: We show that the given $\mathbf{y}$ is feasible, then we'll show that (using the two given solutions) $z = w$ which will make $\mathbf{y}$ optimal for the dual.

- To show that $\mathbf{y}$ is feasible, we need to show that $A^T\mathbf{y} \geq \mathbf{c}$.

  Writing this out, we want:
  $$A^T(\mathbf{c}_B^T B^{-1})^T \geq \mathbf{c}$$

  The expression on the RHS looks familar- Looking at it more closely,

  $$A^T(\mathbf{c}_B^T B^{-1})^T = A^T(B^{-1})^T\mathbf{c}_B = \left(\mathbf{c}_B^T B^{-1} A\right)^T \geq \mathbf{c}$$

  Now we can finish it up, and present a proper proof:

  From the optimality conditions given, we know that the Row 0 coefficients of the optimal primal tableau are all greater than or equal to 0:

  $$-\mathbf{c}^T + \mathbf{c}_B^T B^{-1} A \geq \vec{0}$$

  Therefore, simplifying this, we have:

  $$\mathbf{c}_B^T B^{-1} A \geq \mathbf{c}^T \quad \Rightarrow \quad A^T(\mathbf{c}_B^T B^{-1})^T \geq \mathbf{c}$$

  or, $A^T\mathbf{y} \geq \mathbf{c}$. We have shown that $\mathbf{y}$ is feasible.

- We now know that $\mathbf{y}$ is feasible. Now we show that $z = w$:

  For the primal, z is determined by the RHS of the optimal tableau, which we know (from 6.2) is:
  $$z = \mathbf{c}_B^T B^{-1}\mathbf{b} = \mathbf{y}^T\mathbf{b} = w$$

We have now proven the Dual Theorem, and at the same time we can consider the following Lemma:

Lemma:

(A remark in the book): A basis gives a feasible solution to the primal iff $\mathbf{c}_B^T B^{-1}$ is feasible for the dual.

This gives me a way of **tying** the solutions together!

Implementing the Dual Theorem

We have found that:

- Row 0 in the optimal tableau is given by

$$-\mathbf{c}^T + \mathbf{c}_B^T B^{-1} A$$

- We talked about what the optimal Row 0 coefficients are in each of the following cases (if not a basic variable):
  - For a slack variable, s_i, the Row 0 coefficient is $\mathbf{c}_B^T (B^{-1})_i$.
  - For an excess variable, e_i, the Row 0 coefficient is $-\mathbf{c}_B^T (B^{-1})_i$.
  - For an artificial variable a_i, the Row 0 coefficient (using big M) is $\mathbf{c}_B^T (B^{-1})_i - M$
- Each dual variable is associated with a constraint from the primal.
- Each dual variable is then associated to either a slack variable (s_i for a $\leq$ constraint), an excess variable (e_i for a $\geq$ constraint) or an artificial variable (a_i for an $=$ constraint) from the primal.
- Put the list of three cases together with the fact that

$$\mathbf{y}^T = \mathbf{c}_B^T B^{-1}$$

and we get the following rules for constructing the optimal solution for the dual from the optimal solution to the primal. Denote by s_i^*, e_i^*, a_i^* as the (optimal) Row 0 coefficients for variables s_i, e_i, a_i respectively. For each y_i of the dual, locate the corresponding constraint.

  - If the constraint has slack s_i,
  $$y_i = s_i^*$$
  - If the constraint has excess e_i,
  $$y_i = -e_i^*$$
  - If the constraint is $=$ and has artificial variable a_i, then
  $$y_i = a_i^* - M$$

  Important Note: If we go from the dual to the primer, the first two points are still valid. In the last one, add M instead of subtracting it.