The Dual Theorem (6.7)

Let $\mathcal{B}$ be an optimal basis for the primal. Then

$$\mathbf{y} = (\mathbf{c}_{\mathcal{B}}^T B^{-1})^T$$

is an optimal solution to the dual.

Row 0 for optimal: $-\mathbf{c}^T + \mathbf{c}_B^T B^{-1} A$.

- Slack s_i coeff in Row 0 w/orig col $\mathbf{e}_j$:

$$-0 + \mathbf{c}_B^T B^{-1} \mathrm{e}_j = \mathbf{c}_B^T (B^{-1})_j = y_j$$

- Excess e_i coeff in Row 0 is

$$-0 + \mathbf{c}_B^T B^{-1} (-\mathbf{e}_j) = -\mathbf{c}_B^T (B^{-1})_j = -y_j$$

- Art var a_i, coeff in Row 0 (using big M)

$$M + \mathbf{c}_B^T B^{-1} \mathbf{e}_j = M + \mathbf{c}_B^T (B^{-1})_j = M + y_j$$

Original and final tableau for a maximization problem:

x_1	x_2	s_1	s_2	rhs
-2	1	0	0	0
1	-1	1	0	1
2	1	0	1	6

x_1	x_2	s_1	s_2	rhs
0	0	$4/3$	$1/3$	$10/3$
1	0	$1/3$	$1/3$	$7/3$
0	1	$-2/3$	$1/3$	$4/3$

Solution to the primal:$x_1 = 7/3, x_2 = 4/3$ with $z = 10/3$.
Solution to the dual: $y_1 = 4/3, y_2 = 1/3$ with

$$w = (1)(4/3) + (6)(1/3) = 10/3$$

For practice, let's compute sensitivity on b_1 (the RHS of first constraint).

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x_1	x_2	s_1	s_2	rhs
0	0	$4/3$	$1/3$	$10/3$
1	0	$1/3$	$1/3$	$7/3$
0	1	$-2/3$	$1/3$	$4/3$

The new RHS is $B^{-1}\mathbf{b} + \Delta(B^{-1})_1$, or:

$$\begin{bmatrix} 7/3 \\ 4/3 \end{bmatrix} + \Delta \begin{bmatrix} 1/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} (7+\Delta)/3 \\ (4-2\Delta)/3 \end{bmatrix} \geq 0 \Rightarrow -2 \leq \Delta \leq 7$$

What is the shadow price of the first constraint?

$$2\left(\frac{7+\Delta}{3}\right) - \left(\frac{4-2\Delta}{3}\right) = \frac{10}{3} + \frac{4}{3}\Delta$$

Answer: $4/3$ (which is also y_1)...

What have we done today?

- The Dual Theorem says $\mathbf{y} = (\mathbf{c}_B^T B^{-1})^T$
- Shortcuts for computing this by using the final tableau (the entries of Row 0 in the slack variables)
- There seems to be a relationship between the solutions to the dual and the shadow prices.
  Lemma 5: (A remark in the book): A basis giving a feasible solution is optimal iff $\mathbf{c}_B^T B^{-1}$ is feasible for the dual.