6.9: Duality and Sensitivity Analysis

In 6.8, we saw that the optimal solution for the dual gives the shadow prices for the primal. In 6.9, we see how sensitivity can help us with:

- Changing the objective function coefficient of a NBV.
- Changing the column of a NBV.
- Adding a new "activity" (column).

To illustrate the results, we'll use the following simple primal-dual in normal form:

$$\begin{array}{rl} \max\ z & = \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & A\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{rl} \min\ w & = \mathbf{b}^T\mathbf{y} \\ \text{s.t.} & A^T\mathbf{y} \ge \mathbf{c} \\ & \mathbf{y} \ge 0 \end{array}$$

The three items above correspond to changes that we can track in the primal and dual. We'll list them in general here, then we'll look at specific computational examples below.

- If we change c_i corresponding to a non-basic variable in the primal, that would change only a right-side value c_i in the dual. Therefore, one might imagine that if the corresponding constraint in the dual is satisfied, then the current solution remains optimal (and that is true).
- Similarly, if the column of a non-basic variables changes, that changes the coefficients of a single constraint in the dual. Again, if the constraint remains satisfied, then the current solution remains the same.
- Similarly, adding a new activity corresponds to a new column in the primal, or an extra constraint in the dual. And if the constraint is satisfied in the dual, then the current solution remains optimal for the primal.

And here is a primal problem, with the initial and final tableaux:

$$\begin{array}{rl} \max z = & 3x_1 + 5x_2 \\ \text{st} & 2x_1 + 3x_2 \le 25 \\ & x_1 + 2x_2 \le 5 \\ & x_1, x_2 \ge 0 \end{array} \quad \Rightarrow \quad \begin{array}{rrrr|r} -3 & -5 & 0 & 0 & 0 \\ \hline 2 & 3 & 1 & 0 & 25 \\ 1 & 2 & 0 & 1 & 5 \end{array} \quad \Rightarrow \quad \begin{array}{rrrr|r} 0 & 1 & 0 & 3 & 15 \\ \hline 0 & -1 & 1 & -2 & 15 \\ 1 & 2 & 0 & 1 & 5 \end{array}$$

Here we go!

- Recall how we can find the solution to the dual, given the optimal Row 0. In this case, $\mathbf{y}^T = [0, 3]$.
- Change c_i if x_i is non-basic:

  Initially, we said that this was simple: Suppose $\hat{c}_i$ is the value in Row 0 for the final tableau. Then $\hat{c}_i - \Delta > 0$ is what we need.

  Changing c_i in the objective function will change the i^{th} constraint of the dual.

$$a_{1i}y_1 + a_{2i}y_2 + \cdots + a_{mi}y_m \ge c_i$$

Now, as long as the optimal value satisfies this requirement, then $\mathbf{y}$ remains feasible in the dual.

Example: Now, if we change the coefficient for x_2 from 5 to $5+\Delta$, the corresponding change in the dual would be:

$$3y_1 + 2y_2 \geq c_2 \quad \Rightarrow \quad 3(0) + 2(3) \geq 5 + \Delta$$

Therefore, as long as $\Delta \leq 1$, the current basis remains optimal.

- Change the column corresponding to a non-basic variable.

  In this case, the **column** values of $a_{.,i}$ change and the value of c_j changes, which corresponds to a change ONLY in the i^{th} constraint:

  $$a_{1i}y_1 + a_{2i}y_2 + \cdots + a_{mi}y_m \geq c_i$$

  If it is satisfied, the current basis remains optimal.

  Example:

  Using the previous example again, suppose that column for x_2, which is $[-5, 3, 2]^T$ changes to $[-2, 1, 1]$. Is the current basis still optimal?

  Solution: This means that the second constraint:

  $$(1)(y_1) + (1)(y_2) \geq 2$$

  is satisfied with $y_1 = 0, y_2 = 3$, and it is. Therefore, the current basis does remain optimal.

  Example 2

  We can interpret the column for x_2 in another way- Suppose the amount of resources x_2 takes up (the column $[3, 2]^T$) is fixed. What is it costing us to produce? In other words, how much should we sell x_2 for in order to make it profitable?

  Solution: Assuming the current basis is optimal,

  $$3y_1 + 2y_2 \geq c_2 \quad \Rightarrow \quad 3(0) + 2(3) = 6$$

  So, it is actually costing us \$6 each to make x_2.

- Something similar occurs if we add a new activity- Which means we add a column of variables. And we see that once again, we would simply check that the same constraint is satisfied:

  $$a_{1(n+1)}y_1 + a_{2(n+1)}y_2 + \cdots + a_{m(n+1)}y_m \geq c_{n+1}$$

  Example:

Suppose we want to add a third activity whose column is $[-4, 2, 2]$. Is it worthwhile to bring this in?

Solution: The new constraint would be:

$$2y_1 + 2y_2 \geq 4$$

With the current basis optimal, we have: $2(0) + 2(3) = 6$, which is satisfied. Therefore, it would not be worthwhile to bring this activity in.

Notice that we can use the dual (shadow price) to cost out the value of the resources. In the last example, we saw that, given the resources that were being used, we would have to sell the new thing at at least \$6, otherwise we would lose money. We saw a similar result for x_2.

Just so we can see what happens, if we increase the price for x_2 to 6, here is the new final tableau:

x_1	x_2	s_1	s_2	rhs
0	0	0	3	15
0	-1	1	-2	15
1	2	0	1	5
0	0	0	3	0

And we now see multiple optimal solutions.

- For some extra practice, write down the dual.

  Solution:

$$\begin{aligned} \min\ w \quad &= 25y_1 + 5y_2 \\ \text{s.t.} \quad & 2y_1 + y_2 \geq 3 \\ & 3y_1 + 2y_2 \geq 5 \\ & \mathbf{y} \geq 0 \end{aligned}$$

  Our dual solution is $y_1 = 0, y_2 = 3$, giving $w = 15$ for the optimal value, and $e_1 = 0$ and $e_2 = 1$.