6.10 Complementary Slackness

- This is a useful duality relationship, provable from the Duality Theorem. Consider a primal/dual pair and respective feasible solutions $\mathbf{x}$ and $\mathbf{y}$.
- Recall that we already have one "iff" condition on $\mathbf{x}$ and $\mathbf{y}$ to be optimal solutions of their respective problems (that was, the objective functions are equal, $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$).
- This section gives another condition. Before stating it, recall that a variable in one problem corresponds to a constraint in the other (and vice versa).

We actually looked at complementary slackness in the "motivating slackness" handout. There we saw that:

$$e_j x_j = 0 \qquad \text{and } s_j y_j = 0$$

What does this mean (in words)?

- Suppose the j^{th} constraint of the primal has slack. Would there be any benefit in providing MORE of that resource? (No). Therefore, the shadow price of this constraint should be zero, and that is the value of the dual.
- You might think of this in the opposite way- Suppose the shadow price of a certain constraint is non-negative. What should be true about the slack for that constraint? (It should be zero- You should be using it all up).

A Little Corollary

There is a computation that we can make that comes in handy.[1] Recall our notation for primal-dual:

$$\begin{array}{rl} \max\ z & = \mathbf{c}^T\mathbf{x} \\ \text{st} & A\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{rl} \max\ w & = \mathbf{b}^T\mathbf{y} \\ \text{st} & A^T\mathbf{y} \ge \mathbf{c} \\ & \mathbf{y} \ge 0 \end{array}$$

Given $\mathbf{x}, \mathbf{y}$ feasible for the primal and dual, we will show that

$$w - z = \mathbf{b}^T\mathbf{y} - \mathbf{c}^T\mathbf{x} = \mathbf{y}^T\mathbf{s} + \mathbf{x}^T\mathbf{e}$$

where $\mathbf{s} = \mathbf{b} - A\mathbf{x}$ and $\mathbf{e} = A^T\mathbf{y} - \mathbf{c}$.

NOTE: This is a neat computation in that it tells us how big the gap is between the optimal solutions.

So first, given $\mathbf{x}, \mathbf{y}$ feasible for the primal and dual, we define the slack of the primal and dual as before:

$$\mathbf{s} = \mathbf{b} - A\mathbf{x} \qquad \mathbf{e} = A^T\mathbf{y} - \mathbf{c}$$

We note that $\mathbf{s}, \mathbf{e}$ are both non-negative (due to feasibility). We could write these as:

$$\mathbf{b} = A\mathbf{x} + \mathbf{s} \qquad A^T\mathbf{y} = \mathbf{e} + \mathbf{c}$$

[1] This is not in our text...

Now we're ready:

$$\mathbf{b}^T\mathbf{y} = (A\mathbf{x}+\mathbf{s})^T\mathbf{y} = \mathbf{x}^T A^T\mathbf{y} + \mathbf{s}^T\mathbf{y} = \mathbf{x}^T(\mathbf{e}+\mathbf{c}) + \mathbf{s}^T\mathbf{y}$$

That's it- We just re-write this now:

$$w - z = \mathbf{b}^T\mathbf{y} - \mathbf{c}^T\mathbf{x} = \mathbf{y}^T\mathbf{s} + \mathbf{x}^T\mathbf{e}$$

What happens at optimality?

At optimality, $z = w$ so that the previous expression is just 0:

$$\mathbf{y}^T\mathbf{s} + \mathbf{x}^T\mathbf{e} = \vec{0}$$

From feasibility, all four vectors have non-negative components. That means that the dot products are individually zero as well:

$$\mathbf{y}^T\mathbf{s} = 0 \qquad \mathbf{x}^T\mathbf{e} = 0$$

which gives us our primary result.

Example

Here is one way to use Complementary Slackness to help solve the dual:

- The "Dakota" Problem in the text gives the primal and dual as:

$$\begin{array}{rrrrll}
\max z = & 60x_1 & +30x_2 & +20x_3 & & \\
\text{st} & 8x_1 & +6x_2 & +x_3 & \leq 48 & \text{Lumber} \\
& 4x_1 & +2x_2 & +1.5x_3 & \leq 20 & \text{Finishing} \\
& 2x_1 & +1.5x_2 & +0.5x_3 & \leq 8 & \text{Carpentry} \\
\mathbf{x} \geq 0 & & & & &
\end{array}$$

  and the dual:

$$\begin{array}{rrrrll}
\min w = & 48y_1 & +20y_2 & +8y_3 & & \\
\text{st} & 8y_1 & +4y_2 & +2y_3 & \geq 60 & \text{Desk constraint} \\
& 6y_1 & +3y_2 & +1.5y_3 & \geq 30 & \text{Table constraint} \\
& y_1 & +1.5y_2 & +0.5y_3 & \geq 20 & \text{Chair constraint} \\
\mathbf{y} \geq 0 & & & & &
\end{array}$$

  Suppose we know the optimal solution to the primal,

$$z = 280 \qquad x_1 = 2, \qquad x_2 = 0, \qquad x_3 = 8$$

  And let's see if we can solve the dual.

  SOLUTION: To use complementary slackness, we compare $\mathbf{x}$ with $\mathbf{e}$, and $\mathbf{y}$ with $\mathbf{s}$.

  In looking at $\mathbf{x}$, we see that $e_1 = e_3 = 0$, so those inequality constraints are binding (equality).

Next, computing $\mathbf{s}$, we have

$$\begin{aligned} s_1 = &\quad 48 - (8(2) + 6(0) + 1(8)) = 24 \\ s_2 = &\quad 20 - (4(2) + 2(0) + 1.5(8) = 0 \\ s_3 = &\quad 8 - (2(2) + 1.5(0) + 0.5(8)) = 0 \end{aligned}$$

Then complementary slackness implies that $y_1 = 0$. This, with equality in constraints 1 and 3 gives the 2×2 system:

$$\begin{aligned} 4y_2 + 2y_3 &= 60 \\ 1.5y_1 + 0.5y_2 &= 20 \end{aligned} \quad \Rightarrow \quad y_2 = 10, \quad y_3 = 10$$

And we can verify this is the solution: $w = 48(0) + 20(10) + 8(10) = 280$.

Example 2

Suppose that we have a max in normal form with $\mathbf{c} = [2, 4, 3, 1]$ and

$$A = \begin{bmatrix} 3 & 1 & 1 & 4 \\ 1 & -3 & 2 & 3 \\ 2 & 1 & 3 & -1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 12 \\ 7 \\ 10 \end{bmatrix}$$

Suppose that $x_1 = 0, x_2 = 10.4$, $x_3 = 0$ and $x_4 = 0.4$, and $z = 42$.

Find the solution to the dual using Complementary Slackness. Hint: The first and third constraints of the primal are binding. The second is not.

SOLUTION: Since x_2, x_4 are not zero, the corresponding constraints in the dual are binding (these are constraints 2 and 4 in $A^T\mathbf{y} = \mathbf{c}$).

Since the second constraint in the primal is not binding, then the second dual is zero: $y_2 = 0$.

Put these together, and we have a simple 2×2 system:

$$\begin{aligned} y_1 + y_3 &= 4 \\ 4y_1 - y_3 &= 1 \end{aligned} \quad \Rightarrow \quad y_1 = 1, y_3 = 3$$

Example 3

Consider the following initial and final tableaux, and write the solution to the primal and dual:

x_1	x_2	s_1	s_2	rhs
-2	-2	0	0	0
1	1	1	0	6
2	1	0	1	13

$\Rightarrow$

x_1	x_2	s_1	s_2	rhs
0	0	2	0	12
1	1	1	0	6
0	-1	-2	1	1

Therefore,

x_1	x_2	s_1	s_2
6	0	0	1
e_1	e_2	y_1	y_2
0	0	2	0

Moral: It is possible, when looking at $s_j y_j = 0$ or $e_j x_j = 0$, that BOTH values are zero.