Consider a minimization problem and how we solve it currently:

$$\begin{aligned} \min \quad & w = y_1 + 2y_2 \\ \text{st} \quad & y_1 - 2y_2 + y_3 \geq 4 \\ & 2y_1 + y_2 - y_3 \geq 6 \\ & y_1, y_2, y_3 \geq 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max \quad & w = -y_1 - 2y_2 \\ \text{st} \quad & y_1 - 2y_2 + y_3 \geq 4 \\ & 2y_1 + y_2 - y_3 \geq 6 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

y_1	y_2	y_3	e_1	e_2	rhs
1	2	0	0	0	0
1	-2	1	-1	0	4
2	1	-1	0	-1	6

How to proceed?

What if we do this:

$$\begin{array}{ccccc|c} y_1 & y_2 & y_3 & e_1 & e_2 & rhs \\ 1 & 2 & 0 & 0 & 0 & 0 \\ \hline 1 & -2 & 1 & -1 & 0 & 4 \\ 2 & 1 & -1 & 0 & -1 & 6 \end{array} \rightarrow \begin{array}{ccccc|c} y_1 & y_2 & y_3 & e_1 & e_2 & rhs \\ 1 & 2 & 0 & 0 & 0 & 0 \\ \hline -1 & 2 & -1 & 1 & 0 & -4 \\ -2 & -1 & 1 & 0 & 1 & -6 \end{array}$$

PROBLEM: The Simplex Method must begin with a BFS. This is basic, but not feasible.

However, consider the dual:

$$\begin{array}{rl} \min & w = y_1 + 2y_2 \\ \text{st} & y_1 - 2y_2 + y_3 \geq 4 \\ & 2y_1 + y_2 - y_3 \geq 6 \\ & y_1, y_2, y_3 \geq 0 \end{array} \quad \Rightarrow \quad \begin{array}{rrl} \max & z = 4x_1 + 6x_2 & \\ \text{st} & x_1 + 2x_2 & \leq 1 \\ & -2x_1 + x_2 & \leq 2 \\ & x_1 - x_2 & \leq 0 \\ & x_1, x_2 \geq 0 & \end{array}$$

And the tableaux:

y_1	y_2	y_3	e_1	e_2	rhs
1	2	0	0	0	0
-1	2	-1	1	0	-4
-2	-1	1	0	1	-6

x_1	x_2	s_1	s_2	s_3	rhs
-4	-6	0	0	0	0
1	2	1	0	0	1
2	1	0	1	0	2
-1	-1	0	0	1	0

The tableau on the right: We have a BFS.
The tableau on the left: Basic, but not feasible.

First step in the Simplex method:
Column w/largest negative # Row 0.

x_1	x_2	s_1	s_2	s_3	rhs
-4	-6	0	0	0	0
1	2	1	0	0	1
2	1	0	1	0	2
-1	-1	0	0	1	0

Second step, Simplex: “ratio test” for pivot row.

x_1	x_2	s_1	s_2	s_3	*rhs*
-4	-6	0	0	0	0
1	2	1	0	0	1
2	1	0	1	0	2
-1	-1	0	0	1	0

x_1	x_2	s_1	s_2	s_3	rhs
-4	-6	0	0	0	0
1	2	1	0	0	1
2	1	0	1	0	2
-1	-1	0	0	1	0

$\rightarrow$

y_1	y_2	y_3	e_1	e_2	rhs
1	2	0	0	0	0
-1	2	-1	1	0	-4
-2	-1	1	0	1	-6

Largest negative, Row 0 (Primal) $\rightarrow$ Largest negative, RHS (Dual)

x_1	x_2	s_1	s_2	s_3	rhs
-4	-6	0	0	0	0
1	2	1	0	0	1
2	1	0	1	0	2
-1	-1	0	0	1	0

$\rightarrow$

y_1	y_2	y_3	e_1	e_2	rhs
1	2	0	0	0	0
-1	2	-1	1	0	-4
-2	-1	1	0	1	-6

(Primal) Ratio of RHS to pos col value, take the min.

(Dual) Ratio of Row 0 to neg row value, take min (abs value)

Row Reduce both Primal and Dual

x_1	x_2	s_1	s_2	s_3	
-1	0	3	0	0	3
$1/2$	1	$1/2$	0	0	$1/2$
$-5/2$	0	$-1/2$	1	0	$3/2$
$3/2$	0	$1/2$	0	1	$1/2$

y_1	y_2	y_3	e_1	e_2	*rhs*
0	$3/2$	$1/2$	0	$1/2$	-3
0	$5/2$	$-3/2$	1	$-1/2$	-1
1	$1/2$	$1/2$	0	$-1/2$	3

The current basic solution is:

s_1	s_2	s_3	x_1	x_2	
0	$3/2$	$1/2$	0	$1/2$	Feasible, not optimal
y_1	y_2	y_3	e_1	e_2	
3	0	0	-1	0	Basic, Not feasible

Next step?

Compare Simplex on Left to the Dual on Right:

x_1	x_2	s_1	s_2	s_3	
-1	0	3	0	0	3
$1/2$	1	$1/2$	0	0	$1/2$
$-5/2$	0	$-1/2$	1	0	$3/2$
$3/2$	0	$1/2$	0	1	$1/2$

y_1	y_2	y_3	e_1	e_2	rhs
0	$3/2$	$1/2$	0	$1/2$	-3
0	$5/2$	$-3/2$	1	$-1/2$	-1
1	$1/2$	$1/2$	0	$-1/2$	3

Ratio test gives:

x_1	x_2	s_1	s_2	s_3	
-1	0	3	0	0	3
$1/2$	1	$1/2$	0	0	$1/2$
$-5/2$	0	$-1/2$	1	0	$3/2$
$3/2$	0	$1/2$	0	1	$1/2$

y_1	y_2	y_3	e_1	e_2	rhs
0	$3/2$	$1/2$	0	$1/2$	-3
0	$5/2$	$-3/2$	1	$-1/2$	-1
1	$1/2$	$1/2$	0	$-1/2$	3

Now pivot:

Row Reduce:

x_1	x_2	s_1	s_2	s_3	
0	0	10/3	0	2/3	10/3
0	1	1/3	0	$-1/3$	1/3
0	0	1/3	1	5/3	7/3
1	0	1/3	0	2/3	1/3

y_1	y_2	y_3	e_1	e_2	*rhs*
0	7/3	0	1/3	1/3	$-10/3$
0	$-5/3$	1	$-2/3$	1/3	2/3
1	$-1/3$	0	$-1/3$	$-1/3$	10/3

The current basic solution is:

y_1	y_2	y_3	e_1	e_2	
10/3	0	2/3	0	0	Feasible
s_1	s_2	s_3	x_1	x_2	
0	7/3	0	1/3	1/3	Feasible

Therefore optimal.

Remark 1: Back at the Beginning

Given a min, convert to a max, construct the tableau.
We assume initial tableau Row 0 ≥ 0. Why?

y_1	y_2	y_3	e_1	e_2	rhs
1	2	0	0	0	0
-1	2	-1	1	0	-4
-2	-1	1	0	1	-6

x_1	x_2	s_1	s_2	s_3	rhs
-4	-6	0	0	0	0
1	2	1	0	0	1
2	1	0	1	0	2
-1	-1	0	0	1	0

Row 0 $\geq 0 \Rightarrow$ "Dual" feasible.

Remark 2

In performing ratio test, use negative values.

y_1	y_2	y_3	e_1	e_2	rhs
1	2	0	0	0	0
−1	2	−1	1	0	−4
−2	−1	1	0	1	−6

What if ratio test fails? In primal? (Unbdd)
Example for dual:

y_1	y_2	y_3	e_1	e_2	rhs
1	2	0	0	0	0
−1	2	−1	1	0	−4
2	1	1	0	1	−6

Conclusion?The problem is infeasible: $2y_1 + y_2 + y_3 + e_2 = -6$ but $y_1, y_2, y_3, e_2 \geq 0$.

Dual Simplex Method can be used for this kind of problem:

y_1	y_2	y_3	e_1	e_2	rhs
1	2	0	0	0	0
-1	2	-1	1	0	-4
-2	-1	1	0	1	-6

The Dual Simplex Method (Min)

Convert to max, construct tableau.

1. Assume Row $0 \geq 0$ (Remark 1)
2. Each excess variable has -1 in a particular row. Multiply those by -1.
3. Find Row with largest negative RHS. (No negatives? Optimal)
4. For each non-negative entry in Row 0, abs ratio test with negative values in Pivot Row (denominator). (Can't do it? Infeasible. Remark 2)
5. Pivot.
6. Repeat from Step 3.

Example

Solve the min problem:

$$\begin{aligned} \min \quad & 2x_1 + 3x_2 + 4x_3 \\ \text{st} \quad & x_1 + 2x_2 + x_3 \geq 3 \\ & 2x_1 - x_2 + 3x_3 \geq 4 \end{aligned}$$

SOLUTION: Convert to max, write the tableau, multiply constraints by -1.

x_1	x_2	x_3	e_1	e_2	rhs
2	3	4	0	0	0
1	2	1	-1	0	3
2	-1	3	0	-1	4

$\rightarrow$

x_1	x_2	x_3	e_1	e_2	rhs
2	3	4	0	0	0
-1	-2	-1	1	0	-3
-2	1	-3	0	1	-4

Determine the first pivot position...

Bring in x_1, perform Row Ops:

x_1	x_2	x_3	e_1	e_2	rhs
0	4	1	0	1	-4
0	$-5/2$	$1/2$	1	$-1/2$	-1
1	$-1/2$	$3/2$	0	$-1/2$	2

Determine next pivot... Bring in x_2:

x_1	x_2	x_3	e_1	e_2	rhs
0	0	$9/5$	$8/5$	$1/5$	$-28/5$
0	1	$-1/5$	$-2/5$	$1/5$	$2/5$
1	0	$7/5$	$-1/5$	$-2/5$	$11/5$

Optimal Solution:

$x_1 = 11/5, x_2 = 2/5$, $x_3 = 0$ and $z = 28/5$.

What happens if we want to bring in a new constraint, like $x_1 + 2x_2 \geq 4$?

x_1	x_2	x_3	e_1	e_2	e_3	rhs
0	0	9/5	8/5	1/5	0	−28/5
0	1	−1/5	−2/5	1/5	0	2/5
1	0	7/5	−1/5	−2/5	0	11/5
1	2	0	0	0	−1	4

x_1	x_2	x_3	e_1	e_2	e_3	rhs
0	0	9/5	8/5	1/5	0	−28/5
0	1	−1/5	−2/5	1/5	0	2/5
1	0	7/5	−1/5	−2/5	0	11/5
0	0	1	−1	0	1	−1

Conclusion?

Current solution not optimal. (Check constraint as well)

Note that you could indeed bring the last row in as a pivot row and pivot to get a new optimal solution:

x_1	x_2	x_3	e_1	e_2	e_3	rhs
0	0	$17/5$	0	$1/5$	$8/5$	$-36/5$
0	1	$-3/5$	0	$1/5$	$-2/5$	$4/5$
1	0	$6/5$	0	$-2/5$	$-1/5$	$12/5$
0	0	-1	1	0	-1	1

Uses of Dual Simplex: Add a New Constraint

By adding a new constraint, there are three possible outcomes.

- Current solution satisfies new constraint.
  Conclusion: Old solution still optimal.
- Current solution does not satisfy new constraint, but LP is still feasible.
  Use dual simplex to incorporate the new constraint into tableau and get new solution.
- Current solution does not satisfy new constraint, new LP becomes infeasible.

Uses of Dual Simplex: New RHS, New Solution

- If the RHS of a constraint is changed until it is negative, the solution is no longer feasible.
- To find the new solution, we had to start from the beginning again.
- Now, use Dual Simplex to incorporate the new constraint into the problem.

Example: Change the RHS, get new solution.

Starting (max) tableau and final tableau:

$$\begin{array}{rrrr|r} x_1 & x_2 & s_1 & s_2 & rhs \\ -3 & -2 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 4 \\ 2 & 1 & 0 & 1 & 6 \end{array} \quad \rightarrow \quad \begin{array}{rrrr|r} x_1 & x_2 & s_1 & s_2 & rhs \\ 0 & 0 & 1 & 1 & 10 \\ \hline 0 & 1 & 2 & -1 & 2 \\ 1 & 0 & -1 & 1 & 2 \end{array}$$

By how much can $b_1 = 4$ change?

$$B^{-1}\mathbf{b} + \Delta (B^{-1})_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \Delta \begin{bmatrix} 2 \\ -1 \end{bmatrix} \geq 0$$

We find $-1 \leq \Delta \leq 2$. What happens if $\Delta = -2$?

New tableau becomes:

x_1	x_2	s_1	s_2	rhs
0	0	1	1	10
0	1	2	-1	2
1	0	-1	1	2

$\rightarrow$

x_1	x_2	s_1	s_2	rhs
0	0	1	1	8
0	1	2	-1	-2
1	0	-1	1	4

Find the new solution by using the Dual Simplex:

x_1	x_2	s_1	s_2	rhs
0	1	3	0	6
0	-1	-2	1	2
1	1	1	0	2

Example: Mixed Constraints

In this example, we have mixed constraint types and mix regular and dual simplex methods together to solve the tableau.

$$\begin{array}{rrrl} \max\ z = & -x_1 & +5x_2 & \\ \text{st} & 2x_1 & -3x_2 & \geq 1 \\ & x_1 & +x_2 & \leq 3 \\ & x_1, & x_2 & \geq 0 \end{array} \quad \Rightarrow \quad \begin{array}{rrrr|r} x_1 & x_2 & e_1 & s_2 & \\ 1 & -5 & 0 & 0 & 0 \\ \hline 2 & -3 & -1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 3 \end{array}$$

$$\begin{array}{rrrr|r} x_1 & x_2 & e_1 & s_2 & \\ 1 & -5 & 0 & 0 & 0 \\ \hline -2 & 3 & 1 & 0 & -1 \\ 1 & 1 & 0 & 1 & 3 \end{array}$$

Primal and Dual are both infeasible at this point. Try regular simplex (rule of thumb).

Pivot:

x_1	x_2	e_1	s_2	
1	-5	0	0	0
-2	3	1	0	-1
1	1	0	1	3

$\Rightarrow$

x_1	x_2	e_1	s_2	
1	0	0	5	15
-5	0	1	-3	-10
1	1	0	1	3

Dual feasible. Use dual simplex.
(Find the pivot)

x_1	x_2	e_1	s_2	
0	0	$6/5$	$7/5$	3
1	0	$-1/5$	$3/5$	2
0	1	$1/5$	$2/5$	1

Feasible for both primal and dual- Optimal. Write the solutions?

$$x_1 = 2, x_2 = 1 \qquad y_1 = -6/5, y_2 = 7/5$$

Add in a new constraint $x_1 + 3x_2 \leq 3$.$\mathcal{B}$ remain?

- Check constraint: $(2) + 3(1) = 5$, so "Case 2".
- Find the new solution:

x_1	x_2	e_1	s_2	s_3	
0	0	6/5	7/5	0	3
1	0	−1/5	3/5	0	2
0	1	1/5	2/5	0	1
1	3	0	0	1	3

x_1	x_2	e_1	s_2	s_3	
0	0	6/5	7/5	0	3
1	0	−1/5	3/5	0	2
0	1	1/5	2/5	0	1
0	0	−2/5	−9/5	1	−2

Next pivot: $(3, 4)$

After pivot:

x_1	x_2	e_1	s_2	s_3	
0	0	8/9	0	7/9	13/9
1	0	−1/3	0	1/3	4/3
0	1	1/9	0	2/9	5/9
0	0	2/9	1	−5/9	10/9