A Worked Example in Maple, Big M

(The same example is worked out on the next page in Mathematica)

$$\begin{array}{rrrrl} \min z = & 4x_1 & +x_2 & & \\ \text{st} & 3x_1 & +x_2 & = 3 \\ & 4x_1 & +3x_2 & \geq 6 \\ & x_1 & +2x_2 & \leq 4 \end{array} \quad \Rightarrow \quad \begin{array}{rrrrrr|r} x_1 & x_2 & e_1 & s_1 & a_1 & a_2 & \\ \hline 4 & 1 & 0 & 0 & M & M & 0 \\ \hline 3 & 1 & 0 & 0 & 1 & 0 & 3 \\ 4 & 3 & -1 & 0 & 0 & 1 & 6 \\ 1 & 2 & 0 & 1 & 0 & 0 & 4 \end{array}$$

(all variables non-negative)

```
with(LinearAlgebra):
A:=<<4,3,4,1>|<1,1,3,2>|<0,0,-1,0>|<0,0,0,1>|<M,1,0,0>|<M,0,1,0>|<0,3,6,4>>;
```

1. Step 1: Clean up Row 0.

   ```
   A1:=RowOperation(A,[1,2],-M):
   A2:=RowOperation(A1,[1,3],-M);
   ```

   $$\begin{array}{cccccc|r} 4-7M & 1-4M & M & 0 & 0 & 0 & -9M \\ \hline 3 & 1 & 0 & 0 & 1 & 0 & 3 \\ 4 & 3 & -1 & 0 & 0 & 1 & 6 \\ 1 & 2 & 0 & 1 & 0 & 0 & 4 \end{array}$$

2. Standard simplex: Pivot in first column, first constraint.

   ```
   A3:=RowOperation(A2,2,1/3):
   A4:=RowOperation(A3,[1,2],-4+7*M):
   A5:=RowOperation(A4,[3,2],-4):
   A6:=RowOperation(A5,[4,2],-1);
   ```

   $$\begin{array}{cccccc|r} 0 & \frac{1}{3}-\frac{5}{3}M & M & 0 & -\frac{4}{3}+\frac{7}{3}M & 0 & -4-2M \\ \hline 1 & 1/3 & 0 & 0 & 1/3 & 0 & 1 \\ 0 & 5/3 & -1 & 0 & -4/3 & 1 & 2 \\ 0 & 5/3 & 0 & 1 & -1/3 & 0 & 3 \end{array}$$

3. For the Ratio test, `evalf([1/(1/3), 2/(5/3),3/(5/3)]);`.

   Pivot in constraint 2 (Row 3):

   ```
   A7:=RowOperation(A6,3,3/5):
   A8:=RowOperation(A7,[1,3],1/3+(5/3)*M):
   A9:=RowOperation(A8,[2,3],-1/3):
   A10:=RowOperation(A9,[4,3],-5/3);
   ```

   $$\begin{array}{cccccc|r} 0 & 0 & -1/5 & 0 & -\frac{8}{5}+M & \frac{1}{5}+M & -18/5 \\ \hline 1 & 0 & 1/5 & 0 & 3/5 & -1/5 & 3/5 \\ 0 & 1 & -3/5 & 0 & -4/5 & 3/5 & 6/5 \\ 0 & 0 & 1 & 1 & 1 & -1 & 1 \end{array}$$

4. STOP: Notice that at this stage, we have a feasible solution to the original problem (it is not optimal yet). we can either remove or ignore the artificial variables. To continue as usual, pivot in the third column, third constraint (Row 4):

   ```
   A11:=RowOperation(A10,[1,4],1/5):
   A12:=RowOperation(A11,[2,4],-1/5):
   A13:=RowOperation(A12,[3,4],3/5);
   ```

   $$\begin{array}{cccccc|r} 0 & 0 & 0 & 1/5 & * & * & -17/5 \\ \hline 1 & 0 & 0 & -1/5 & * & * & 2/5 \\ 0 & 1 & 0 & 3/5 & * & * & 9/5 \\ 0 & 0 & 1 & 1 & * & * & 1 \end{array}$$

   Be sure to summarize the solution: “The original function is minimized when $x_1 = 2/5, x_2 = 9/5$, and the optimal value is 17/5.”

A Worked Example in Mathematica, Big M

$$\begin{array}{rlll} \min z = & 4x_1 & +x_2 & \\ \text{st} & 3x_1 & +x_2 & = 3 \\ & 4x_1 & +3x_2 & \geq 6 \\ & x_1 & +2x_2 & \leq 4 \end{array} \quad \Rightarrow \quad \begin{array}{cccccc|c} x_1 & x_2 & e_1 & s_1 & a_1 & a_2 & \\ \hline 4 & 1 & 0 & 0 & M & M & 0 \\ \hline 3 & 1 & 0 & 0 & 1 & 0 & 3 \\ 4 & 3 & -1 & 0 & 0 & 1 & 6 \\ 1 & 2 & 0 & 1 & 0 & 0 & 4 \end{array}$$

(all variables non-negative)

```
A = {{4, 1, 0, 0, M, M, 0}, {3, 1, 0, 0, 1, 0, 3}, {4, 3, -1, 0, 0, 1,6}, {1, 2, 0, 1, 0, 0, 4}};
A // MatrixForm;
```

1. Step 1: Clean up Row 0.

```
A1 = A;
A1[[1]] = -M*A[[2]] + A[[1]];
A1[[1]] = -M*A[[3]] + A1[[1]];
A1 // MatrixForm
```

$$\begin{array}{cccccc|c} 4-7M & 1-4M & M & 0 & 0 & 0 & -9M \\ \hline 3 & 1 & 0 & 0 & 1 & 0 & 3 \\ 4 & 3 & -1 & 0 & 0 & 1 & 6 \\ 1 & 2 & 0 & 1 & 0 & 0 & 4 \end{array}$$

2. Start with column 1, then perform the Ratio Test to find the pivot row.

```
Ratio1 = {3/3, 6/4, 4/1}

A2 = A1;
A2[[2]] = (1/3)*A1[[2]];
A2[[1]] = (-4 + 7*M)*A2[[2]] + A1[[1]];
A2[[1]] = Simplify[A2[[1]]];
A2[[3]] = -4*A2[[2]] + A1[[3]];
A2[[4]] = -1*A2[[2]] + A1[[4]];
A2 // MatrixForm
```

$$\begin{array}{cccccc|c} 0 & \frac{1}{3}-\frac{5}{3}M & M & 0 & -\frac{4}{3}+\frac{7}{3}M & 0 & -4-2M \\ \hline 1 & 1/3 & 0 & 0 & 1/3 & 0 & 1 \\ 0 & 5/3 & -1 & 0 & -4/3 & 1 & 2 \\ 0 & 5/3 & 0 & 1 & -1/3 & 0 & 3 \end{array}$$

3. For the Ratio test,

```
Ratio2 = {A4[[2, 7]]/A4[[2, 2]], A4[[3, 7]]/A4[[3, 2]], A4[[4, 7]]/A4[[4, 2]]}

A3 = A2;
A3[[3]] = (3/5)*A3[[3]];
A3[[1]] = A3[[1]] + (1/3)*(1 + 5*M)*A3[[3]];
A3[[1]] = Simplify[A3[[1]]];
A3[[2]] = A3[[2]] - (1/3)*A3[[3]];
A3[[4]] = A3[[4]] - (5/3)*A3[[3]];
A3 // MatrixForm
```

$$\begin{array}{cccccc|c} 0 & 0 & -1/5 & 0 & -\frac{8}{5}+M & \frac{1}{5}+M & -18/5 \\ \hline 1 & 0 & 1/5 & 0 & 3/5 & -1/5 & 3/5 \\ 0 & 1 & -3/5 & 0 & -4/5 & 3/5 & 6/5 \\ 0 & 0 & 1 & 1 & 1 & -1 & 1 \end{array}$$

4. STOP: Notice that at this stage, we have a feasible solution to the original problem (it is not optimal yet). we can either remove or ignore the artificial variables. To continue as usual, pivot in the third column, third constraint (Row 4):

```
A4 = A3;
A4[[3]] = (3/5)*A4[[4]] + A4[[3]];
A4[[2]] = -(1/5)*A4[[4]] + A4[[2]];
A4[[1]] = (1/5)*A4[[4]] + A4[[1]];
A4 // MatrixForm
```

$$\begin{array}{cccccc|c} 0 & 0 & 0 & 1/5 & * & * & -17/5 \\ \hline 1 & 0 & 0 & -1/5 & * & * & 2/5 \\ 0 & 1 & 0 & 3/5 & * & * & 9/5 \\ 0 & 0 & 1 & 1 & * & * & 1 \end{array}$$

Be sure to summarize the solution: " The original function is minimized when $x_1 = 2/5$, $x_2 = 9/5$, and the optimal value of z is 17/5."

Examples, Big M Method

Try these out. The solutions are on the next page, but don't look at them until you've worked these out yourself!

1. Here is the LP:

$$\begin{array}{rllll} \max z = & -x_1 & +2x_2 & & \\ \text{st} & x_1 & +x_2 & \geq 2 \\ & -x_1 & +x_2 & \geq 1 \\ & & x_2 & \leq 3 \end{array}$$

   Write the initial and final tableau, and interpret the solution:

2. This is the LP:

$$\begin{array}{lrrrl} \max z = & x_1 & +3x_2 & +x_3 & \\ \text{st} & x_1 & +x_2 & +2x_3 & \leq 4 \\ & -x_1 & & +x_3 & \geq 4 \\ & & & x_3 & \geq 3 \end{array}$$

   Write the initial and final tableau and the interpretation:

3. Here is the LP

$$\begin{array}{lrrrrl} \max z = & x_1 & +x_2 & & & \\ \text{st} & x_1 & -x_2 & -x_3 & & = 1 \\ & -x_1 & +x_2 & +2x_3 & -x_4 & = 1 \end{array}$$

   Write the initial and final tableau and the interpretation:

Solutions to the practice problems

Be sure you can work through these using either Maple or Mathematica.

1. Here is the LP:

$$
\begin{array}{rllll}
\max z = & -x_1 & +2x_2 & \\
\text{st} & x_1 & +x_2 & \geq 2 \\
 & -x_1 & +x_2 & \geq 1 \\
 & & x_2 & \leq 3
\end{array}
$$

   Write the initial and final tableau, and interpret the solution:

   SOLUTION:

	x_1	x_2	e_1	e_2	s_1	a_1	a_2	
	1	−2	0	0	0	M	M	0
a_1	1	1	−1	0	0	1	0	2
a_2	−1	1	0	−1	0	0	1	1
s_1	0	1	0	0	1	0	0	3

	x_1	x_2	e_1	e_2	s_1	a_1	a_2	
	1	0	0	0	2	M	M	0
e_2	1	0	0	1	1	0	−1	2
x_2	0	1	0	0	1	0	0	3
e_1	1	0	1	0	1	−1	0	1

2. This is the LP:

$$
\begin{array}{rlllll}
\max z = & x_1 & +3x_2 & +x_3 & \\
\text{st} & x_1 & +x_2 & +2x_3 & \leq 4 \\
 & -x_1 & & +x_3 & \geq 4 \\
 & & & x_3 & \geq 3
\end{array}
$$

   Write the initial and final tableau and the interpretation:

   SOLUTION:

	x_1	x_2	x_3	s_1	e_1	e_2	a_1	a_2	rhs
	−1	−3	−1	0	0	0	M	M	
s_1	1	1	2	1	0	0	0	0	4
a_1	−1	0	1	0	−1	0	1	0	4
a_2	0	0	1	0	0	−1	0	1	3

   Here's the "final tableau":

	x_1	x_2	x_3	s_1	e_1	e_2	a_1	a_2	rhs
	$-\frac{1}{2}+2M$	$-\frac{5}{2}+M$	0	$\frac{1}{2}+M$	M	M	0	0	$2-3M$
x_3	1/2	1/2	1	1/2	0	0	0	0	2
a_1	−3/2	−1/2	0	−1/2	−1	0	1	0	2
a_2	−1/2	−1/2	0	−1/2	0	−1	0	1	1

3. Here is the LP

$$
\begin{array}{rllllll}
\max z = & x_1 & +x_2 & & & \\
\text{st} & x_1 & -x_2 & -x_3 & & =1 \\
 & -x_1 & +x_2 & +2x_3 & -x_4 & =1
\end{array}
$$

   Write the initial and final tableau and the interpretation:

x_1	x_2	x_3	x_4	a_1	a_2	
−1	−1	0	0	M	M	0
1	−1	−1	0	1	0	1
−1	1	2	−1	0	1	1

	x_1	x_2	x_3	x_4	a_1	a_2	
	0	−2	0	−1	$2+M$	$1+M$	3
x_1	1	−1	0	−1	2	1	3
x_3	0	0	1	−1	1	1	2