

Section 7.2

Quick overview today:

- What is a basic feasible solution for the transportation problem? How can one be identified in a transportation tableau?
- Methods for computing an initial BFS:
 - NW Corner Rule
 - Minimum Cost Rule
 - Vogel's Method

The Transportation Problem and its Dual

Reminder:

$$\begin{array}{ll} \max z = \mathbf{c}^T \mathbf{x} & \Leftrightarrow \min w = \mathbf{b}^T \mathbf{y} \\ \text{st } A\mathbf{x} \leq \mathbf{b} & \text{st } A^T \mathbf{y} \geq \mathbf{c} \end{array}$$

Now, suppose we have a small transportation problem given below. Write down its dual:

	C1	C2	C3	
W 1	3	7	6	5
W 2	2	4	3	2
Demand	2	3	2	7

SOLUTION: We can think of the original as the dual, and write the primal. Here's the original:

$$\begin{array}{rcl} \min w = & 3y_{11} & +7y_{12} & +6y_{13} & +2x_{21} & +4x_{22} & +3x_{23} \\ \text{st } & y_{11} & +y_{12} & +y_{13} & & & & = 5 \\ & & & & y_{21} & +y_{22} & +y_{23} & = 2 \\ \hline & y_{11} & & & +y_{21} & & & = 2 \\ & & y_{12} & & & +y_{22} & & = 3 \\ & & & y_{13} & & & +y_{23} & = 2 \end{array}$$

Now we can see the dual. Let u_i be the dual variable for supply, and v_j be a dual variable for the demand. Then we have the dual:

$$\max z = 5u_1 + 2u_2 + 2v_1 + 3v_2 + 2v_3$$

such that:

$$\begin{array}{rcl|l} u_1 & +v_1 & & \leq 3 \\ u_1 & & +v_2 & \leq 7 \\ u_1 & & & +v_3 \leq 6 \end{array} \quad \left| \quad \begin{array}{rcl} u_2 & +v_1 & & \leq 2 \\ u_2 & & +v_2 & \leq 4 \\ u_2 & & & +v_3 \leq 3 \end{array} \right.$$

where u_i, v_j are URS.

NOTE TO SELF: Show students how we might put u_i, v_j on the table from the beginning, and where the c_{ij} are.

The General Case for the Dual

The general dual will be given by:

$$\max z = \sum_{i=1}^m s_i u_i + \sum_{j=1}^n d_j v_j$$

such that

$$c_{ij} - (u_i + v_j) \geq 0$$

where u_i, v_j are URS. Later on, we will show a quick method of computing the dual solution given a transportation tableau with BFS.

For now, let's continue by thinking about what a BFS looks like in the transportation tableau.

A Basic Feasible Solution

Recall that the Fundamental Theorem of Linear Programming stated that if an LP had an optimal solution, then it had an optimal **basic feasible solution**. Therefore, we need to know (i) Do we have a basic feasible solution? and (ii) Is our current solution optimal?

We'll talk about (i) first:

Lemma 1: (Feasibility)

All balanced transportation problems have a feasible solution.

The feasible solution is below (EXERCISE: Students should show this is indeed a feasible solution). Let $\sum_i s_i = \sum_j d_j = T$. Then

$$x_{ij} = \frac{s_i d_j}{T}$$

EXAMPLE: Using our previous 2×3 transportation problem, a feasible solution is:

	C_1	C_2	C_3	
W_1	10/7	15/7	10/7	5
W_2	4/7	6/7	4/7	2
	2	3	2	

NOTE: This may be *feasible*, but it is not *basic*.

Lemma 2: (Rank of A)

Given the matrix A (which has dimensions $(m+n) \times (mn)$), the rank of A is exactly $m+n-1$.

Comment: It is clear from the exercise about summing the supply and demand that the system is NOT full rank, so that implies that the rank of A is less than or equal to $m+n-1$. Going the other direction takes a bit more explanation because it exploits the structure of A (See Corollary 8.2 in Dantzig (proof as exercise), or Luenberger).

Lemma 3: (Independence)

A set of variables will be independent if they form no loop in the transportation tableau. We won't prove this, but it is also exploiting the structure of the matrix A .

Note: Bazaara goes into a detailed description, but uses arguments from his previous chapter about matrices for networks.

In summary, a solution is a basic solution if it has $m+n-1$ assigned variables, and those variables form no loops. It is a degenerate basic solution if one of the assigned variables is 0.

Lemma 4: (The Integral Property)

If the supply and demand quantities are integers, then the basic variables to the transportation problem are integer valued. Some of you may have seen that in linear algebra- All of the submatrices of A have a determinant of ± 1 , so the inverse will have integer entries.

Finding a Basic Feasible Solution

Method 1: The NW Corner Rule

The NW Corner rule begins with x_{11} .

- Allocate as much as you can into the current position.
- Move one column to the right if there is any supply remaining.

- If there is no supply remaining, move one row down.

Here's an example with a transportation tableau, but with three plants and 4 destinations.

	1	2	3	4	
Plant 1	3	7	6	4	5
Plant 2	2	4	3	2	2
Plant 3	4	3	8	8	3
Demand	3	3	2	2	10

If we go to x_{11} , we can put 3 units there. That finishes customer 1, move right. We have 2 units left; put them in x_{12} . That finishes the supply of 5, and we have, and so on. Here is the final tableau.

	1	2	3	4	
Plant 1	3 3	7 2	6	4	5
Plant 2	2	4 1	3 1	2	2
Plant 3	4	3	8 1	8 2	3
Demand	3	3	2	2	10

For future reference, write down the current cost:

$$w = 9 + 14 + 4 + 3 + 8 + 16 = 54$$

Extra Example (Students try this)

	1	2	3	4	
	4	6	8	8	40
	6	8	6	7	
	5	7	6	8	
	20	30	50	50	150

Max out the upper corner- 20 is the largest value. That finishes the first column, and leaves 20 in the first row.

	1	2	3	4	
20	4	6	8	8	20
x	6	8	6	7	60
x	5	7	6	8	50
0	30	50	50	150	

Go to the NW corner and max it out with another 20. That finishes the first row.

	1	2	3	4	
20	4	6	8	8	0
x	6	8	6	7	60
x	5	7	6	8	50
0	10	50	50	150	

Go to the NW corner and max it out with 10. That finishes the second column.

	1	2	3	4	
	4	6	8	8	
	20	20	x	x	0
	6	8	6	7	
	x	10			50
	5	7	6	8	
	x	x			50
	0	0	50	50	150

After we max out the NW corner with 50, cross out only one of the row OR column, but NOT BOTH:

	1	2	3	4	
	4	6	8	8	
	20	20	x	x	0
	6	8	6	7	
	x	10	50		0
	5	7	6	8	
	x	x	x		50
	0	0	0	50	150

How to finish this off? WE PUT IN A ZERO (important!)

	1	2	3	4	
	4	6	8	8	
	20	20	x	x	0
	6	8	6	7	
	x	10	50	0	0
	5	7	6	8	
	x	x	x	50	0
	0	0	0	0	

This is a degenerate solution, and in a degenerate solution, it is important to note what the basic variables are. Some people will use ϵ to denote the zero.

The cost here is $z = 980$.

Minimum Cost Method

Since we did not take any costs into account in the previous problem, we really don't expect that BFS to be any good. As an alternative, here is the minimum cost method:

- Find the cell with the smallest cost.
- Put as large an allocation as possible into that cell, which should deplete either the row or column (or possibly both). Scratch row/column that is depleted. In the case of a tie, choose one (but not both).
- Repeat.

Example 1

	1	2	3	4	
	8	6	10	9	45
	9	12	13	7	60
	14	9	16	5	50
	45	30	40	40	155

The lowest cost is currently 5 in the lower right corner. Max it out, which finishes the last column.

	1	2	3	4	
	8	6	10	9	45
				x	
	9	12	13	7	60
				x	
	14	9	16	5	10
				40	
	45	30	40	0	

The next is 6, in the top of column 2, and that finishes off column 2. The next one after that is 8 in the upper left corner, and that finishes off row one.

	1	2	3	4	
	8	6	10	9	
15		30	x	x	0
	9	12	13	7	
		x		x	60
	14	9	16	5	
		x		40	10
30		0	40	0	

Continuing, 9 is next, and finishes column 1.

	1	2	3	4	
	8	6	10	9	
15		30	x	x	0
	9	12	13	7	
30		x		x	30
	14	9	16	5	
x		x		40	10
0		0	40	0	

Finally, fill in the last two.

	1	2	3	4	
	8	6	10	9	
15		30	x	x	0
	9	12	13	7	
30		x	30	x	0
	14	9	16	5	
x		x	10	40	0
0		0	0	0	

The Min Cost method works slightly better than the NW corner rule in getting us a “good” basic feasible solution, but perhaps with a little more work, we can get a better version. For future reference, current cost is $z = 1400$.

Example 2 (For Students)

	1	2	3	4	
	4	6	8	8	40
	6	8	6	7	
	5	7	6	8	
	20	30	50	50	150

One solution is the following (there are some ties):

	1	2	3	4	
	4	6	8	8	40
20		20			
	6	8	6	7	60
		10		50	
	5	7	6	8	50
		0	50		
	20	30	50	50	150

This was $z = 930$

Vogel's Approximation Method (VAM)

In VAM, we compute "penalties" - That is, the difference between the 2d most minimum and the minimum in each row and column.

	1	2	3	4		
	8	6	10	9		
					45	8-6=2
	9	12	13	7		
					60	9-7=2
	14	9	16	5		
					50	9-5=4
	45	30	40	40	155	
	9-8=1	9-6=3	13-10=3	7-5=2		

Go to the row or column with the maximum penalty, then use “greedy” method (max out the cell with the minimum cost).

	1	2	3	4		
	8	6	10	9		
				x	45	8-6=2
	9	12	13	7		
				x	60	12-9=3
	14	9	16	5		
				40	10	14-9=5
	45	30	40	0	155	
	9-8=1	9-6=3	13-10=3	x		

The max is again the bottom row- Go to the 9 and max it out. Recalculate penalties if necessary.

	1	2	3	4		
	8	6	10	9		
				x	45	8-6=2
	9	12	13	7		
				x	60	12-9=3
	14	9	16	5		
	x	10	x	40	0	x
	45	20	40	0	155	
	9-8=1	12-6=6	13-10=3	x		

Now we go to the second column, top entry and max it out.

	1	2	3	4		
	8	6	10	9		
		20		x	25	10-8=2
	9	12	13	7		
		x		x	60	13-9=4
	14	9	16	5		
	x	10	x	40	0	x
	45	0	40	0	155	
	9-8=1	x	13-10=3	x		

Now we enter the second row.

	1	2	3	4		
	8	6	10	9		
	x	20		x	25	10
	9	12	13	7		
	45	x		x	15	13
	14	9	16	5		
	x	10	x	40	0	x
	0	0	40	0	155	
	x	x	13-10=3	x		

That doesn't leave us with any choices- Fill in the remainder. Cost is 1260.

	1	2	3	4	
	8	6	10	9	
		20	25		45
	9	12	13	7	
	45		15		60
	14	9	16	5	
		10		40	50
	45	30	40	40	155

Example 2 for students

	1	2	3	4	
	19	30	50	10	7
	20	30	50	60	9
	40	8	70	20	18
	5	8	7	14	

Compute the differences, work in Column 2, max out last row of Column 2.

Diff	1	2	3	4	
9	19	30	50	10	7
		x			
10	20	30	40	60	9
		x			
12	40	8	70	20	10
		8			
	5	0	7	14	
	1	22	10	10	

Compute the differences again- Tie between rows 2 and 3- We chose row 2.

Diff	1	2	3	4	
9	19	30	50	10	7
	x	x			
20	20	30	40	60	4
	5	x			
20	40	8	70	20	10
	x	8			
	0	0	7	14	
	1	x	10	10	

Differences again. Row 1 is max. Load up the upper right corner.

Diff	1	2	3	4	
40	19 x	30 x	50 x	10 7	0
20	20 5	30 x	40	60	4
20	40 x	8 8	70	20	10
	0 x	0 x	7 10	7 10	

Differences again. Col 4 is max, so max out lower right cell.

Diff	1	2	3	4	
x	19 x	30 x	50 x	10 7	0
20	20 5	30 x	40	60 x	4
20	40 x	8 8	70	20 7	3
	0 x	0 x	0 30	7 40	

No choices left. Here's the final tableau:

	1	2	3	4	
	19	30	50	10	7
	20	30	50	60	9
	40	8	70	20	18
	5	8	7	14	