Section 8.1

In these sections, we will:

- Define a graph with its parts (8.1 is about 1 page long with no HW)
- Additionally, we'll look at converting problems into graphs.

Introduction to Graphs

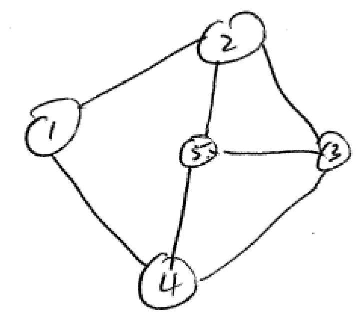
A **graph** (or **network**) is a collection of nodes and edges (or arcs). To be more precise, consider the example below, where the nodes are indexed and the edges will be defined by which nodes they connect as a (possibly ordered) pair. If the graph's edge can only be traversed in one direction, then the graph is said to be a directed graph, and the edges (a, b) and (b, a) would be distinct.

The undirected graph G is defined by two sets of symbols. Set V contains the vertices or nodes, and set E contains the edges or arcs.

$$V = \{1, 2, 3, 4, 5\}$$

$$E = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 2), (1, 4), (3, 5)\}$$

$$G = (V, E)$$

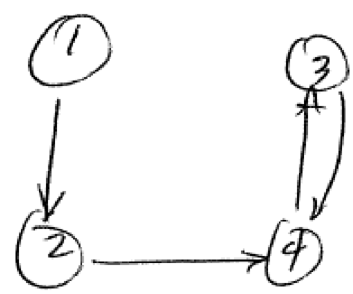


The graphs we will work with typically are directed graphs. Here's an example of a directed graph. The key difference is that the edges are now ordered pairs.

$$V = \{1, 2, 3, 4\}$$

$$E = \{(1, 2), (2, 4), (4, 3), (3, 4)\}$$

$$G = (V, E)$$



Some Definitions

- A **chain** is a sequence of edges (or arcs) such that each edge has exactly one vertex (or node) in common with the previous arc. To clarify,

$$\{(1, 2), (1, 3), (2, 3)\} \text{ is an example of a chain}$$

$$\{(1, 2), (2, 1)\} \text{ is NOT an example.}$$

- A **path** is a chain in which the terminal of the previous edge is the initial node of the next edge. To clarify,

$$\{(1, 2), (2, 3), (3, 4)\} \text{ is an example of a path}$$

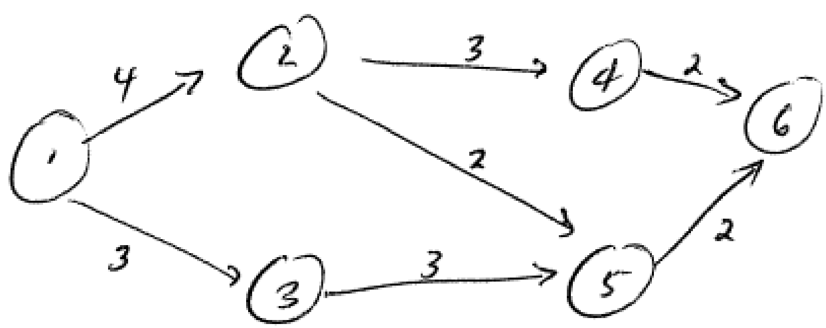
$\{(1,2),(1,3),(2,3)\}$ is NOT an example.

Constructing Graphs

Some situations lend themselves very naturally to a graph. The PowerCo problem is one of them. The other two problems (Bookshelf and Maintenance) we won't solve until later, but its good to consider how to construct them as networks now.

PowerCo

Power is sent from the station at node 1 (the **initial node**) to node 6 (the **terminal node**), and the other nodes are power substations. The costs of transmission are shown on the edges, so for example, it costs \$4.00 per unit to send a unit of power from node 1 to node 2. Our goal is probably to find a path through the graph with the least cost (we'll do this later).



Convert to LP

Before we leave the PowerCo example, it would be worthwhile to tie the graph to our linear programming. The graph suggests a transportation problem, and from that we can get an LP.

As a transportation tableau, we have one supply, one demand, and the remaining nodes are transhipment nodes. Therefore, we get a table like the one below. We don't have a natural supply/demand, so we want to think of sending one unit through node 1 and have a demand of 1 at node 6. That makes $s = 1$ in our transhipment problem, and all supplies and demands are 1. You might also note that we allow transhipment points to ship to themselves at no cost, and that is true here- I've provided the optimal solution in the graph below. Do you see what the utility of allowing nodes to ship to themselves might be? (In this case, they represented unused nodes)

	2	3	4	5	6	Supply
1	4 1	3	M	M	M	1
2	0	M	3	2 1	M	1
3	M	0 1	M	3	M	1
4	M	M	0 1	M	2	1
5	M	M	M	0	2 1	1
Demand	1	1	1	1	1	5

Shelving Units

(First part of Exercise 8, p 419) Build a network diagram from which we could solve the following problem (a hint follows the description):

A library needs to build shelving for:

- 200 books that are 4 inches tall each.
- 100 books that are 8 inches tall each.
- 80 books that are 12 inches tall each.

Each book is 1/2 inch thick. There are several ways the library might store the books- For example, it could make all the shelving 12 inches tall and store all the books there. There is a cost, part of which will depend on the "area" being utilized. There is a fixed cost of \$2300 per shelf, and an additional cost of \$5 per square inch (where the area is height $\times$ 1/2).

For example, what is the cost to store all books using a 12" bookshelf?

$$2300 + 380 \times 5(12 \times 1/2) = 2300 + 1900 \times 6 = 13,700$$

Build a network diagram that would help us minimize cost. HINT: Let C_{ij} be the total cost of shelving all books of height $> i$ and $\leq j$. For example, C_{08} is the cost of shelving books of 4 and 8 inches.

SOLUTION: We compute the costs (the 6 edges):

- Cost for shelving only 4" books:

  $C_{0,4} = 2300 + 200 \cdot 5 \cdot (4 \cdot 1/2) = 4300$

- Cost for 4" and 8" books:

  $C_{0,8} = 2300 + 300 \cdot 5 \cdot (8 \cdot 1/2) = 8300$

- Cost for all books (using 12" shelving):

  $C_{0,12} = 13700$

- Only 8" books:

  $C_{4,8} = 2300 + 100 \cdot 5 \cdot (8 \cdot 1/2) = 4300$

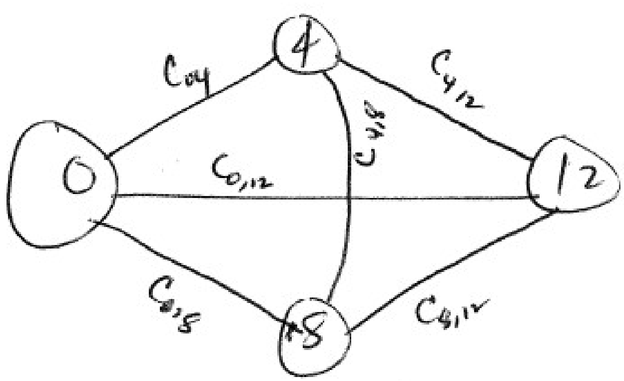
- For 8" and 12":

  $C_{4,12} = 2300 + 180 \cdot 5 \cdot (12 \times 1/2) = 7700$

- Only 12" books:

  $C_{8,12} = 2300 + 80 \cdot 5 \cdot (12 \times 1/2) = 4700$

This yields the network below. We want to find a path from node 0 to node 12.



Machinery - Buy or Maintain?

At the beginning of year 1, a new machine must be purchased. The cost of maintaining the machine i years old is given below, where we also predict the cost of purchasing a new machine (at the beginning of the year).

There is no trade-in value when a machine is replaced. Find a network representation of the problem if our ultimate goal is to minimize the total cost of having a machine for five years.

Year	1	2	3	4	5
Maintenance Cost(*)	38	50	97	182	304
Purchase Cost(**)	170	190	210	250	300

Notes:

(*) The maintenance cost (in thousands of dollars). The maintenance cost for the first year of a new machine is $38,000. Therefore, buying a machine at the beginning and maintaining it for the first year is $170 + 38 = 208$.

(**) The purchase cost (in thousands) is at the beginning of the listed year.

Some other extreme situations:

- Buy the machine and maintain it for the entire time: $170+38+50+97+182+304 = 841$
- Buy a new machine every year:

  $(170 + 38) + (190 + 38) + (210 + 38) + (250 + 38) + (300 + 38) = 1120 + 5(38) = 1310$

Trial Run

We need to know what year to buy and how many years to maintain, so should we take:

$$C_{11} = 170 + 38, \qquad C_{12} = 170 + 38 + 50$$

The problem is that C_{21} should be the edge back from 2 to 1, but that wouldn't make sense.

Better solution

A better solution is to take (i, j) to be a time nodes, where i is the year we buy a new machine (at the beginning of year i), and j is the beginning of the year we buy a replacement machine. Then C_{11} wouldn't make sense, but the remaining values below make sense:

Buy the machine once, then maintain it forever:

$$C_{16} = 841$$

Purchasing one every year:

$$C_{12} + C_{23} + C_{34} + C_{45} + C_{56} = 1310$$

To purchase at year one and maintain for different lengths:

$$\begin{aligned}
C_{12} &= 170 + 38 &&= 208\\
C_{13} &= 170 + 38 + 50 &&= 258\\
C_{14} &= 170 + 38 + 50 + 97 &&= 355\\
C_{15} &= 170 + 38 + 50 + 97 + 182 &&= 537\\
C_{16} &= 170 + 38 + 50 + 97 + 182 + 304 &&= 810
\end{aligned}$$

Similarly,

$$\begin{aligned}
C_{23} &= 190 + 38 &&= 228\\
C_{24} &= 190 + 38 + 50 &&= 278\\
C_{25} &= 190 + 38 + 50 + 97 &&= 375\\
C_{26} &= 190 + 38 + 50 + 97 + 182 &&= 557
\end{aligned}$$

And so on:

$$\begin{aligned} C_{34} = &\ 210 + 38 &= 248 \\ C_{35} = &\ 210 + 38 + 50 &= 298 \\ C_{36} = &\ 210 + 38 + 50 + 97 &= 395 \end{aligned} \qquad \begin{aligned} C_{45} = &\ 210 + 38 &= 248 \\ C_{46} = &\ 210 + 38 + 50 &= 298 \end{aligned}$$

And $C_{56} = 338$.

The network is a bit of a mess, and is very similar to what the book gave for car maintenance- We start at 1 and go to 6.