Review Questions, Final Exam

1. General questions:
   (a) What is the Fundamental Theorem of Linear Programming?
   (b) What is the main idea behind the Simplex Method? (Think about what it is doing graphically- How does the algorithm start, how does it proceed?)

2. Consider the following LP:

$$\begin{aligned} \min z = \quad & 3x - 4y + 2z \\ \text{st} \quad & 2x - 4y \geq 4 \\ & x + z \geq -5 \\ & y + z \leq 1 \\ & x + y + z = 3 \end{aligned}$$

   with $x \geq 0, y$ is URS, $z \geq 0$.
   (a) Write the dual.
   (b) Going back to the original LP, write it in standard form as a max problem with equality constraints, and then write the initial tableau (before big-M or other methods).

3. Consider again the "Wyndoor" company example we looked at in class:

$$\begin{aligned} \min z = \quad & 3x_1 + 5x_2 \\ \text{st} \quad & x_1 \leq 4 \\ & 2x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 18 \end{aligned}$$

   with x_1, x_2 both non-negative.
   (a) Rewrite so that it is in standard form.
   (b) Let s_1, s_2, s_3 be the extra variables introduced in the last answer. Is the following a basic solution? Is it a basic feasible solution?

$$x_1 = 0, x_2 = 6, s_1 = 4, s_2 = 0, s_3 = 6$$

   Which variables are BV, and which are NBV?
   (c) Find the basic feasible solution obtained by taking s_1, s_3 as the non-basic variables.

4. Given the current tableau (with variables labeled above the respective columns), answer the questions below.

x_1	x_2	s_1	s_2	rhs
0	-1	0	2	24
0	$1/3$	1	$-1/3$	1
1	$2/3$	0	$1/3$	4

   (a) Is the tableau optimal (and did your answer depend on whether we are maximizing or minimizing)? For the remaining questions, you may assume we are maximizing.
   (b) Give the current BFS.
   (c) Directly from the tableau, can I increase x_2 from 0 to 1 and remain feasible? Can I increase it to 4?

(d) If x_2 is increased from 0 to 1, compute the new value of z, x_1, s_1 (assuming s_2 stays zero).

(e) Write the objective function and all variables in terms of the non-basic (or free) variables, and then put them in vector form.

5. Given the following final tableau, find two solutions to the original problem.

x_1	x_2	x_3	s_1	s_2	rhs
0	0	5	0	1	15
0	2/5	9/5	1	−1/5	3
1	3/5	6/5	0	1/5	3

3. Set up the initial tableau for the big-M method, and state what your first step would be.

$$\begin{aligned} \max z = \quad & 5x_1 - x_2 \\ \text{st} \quad & 2x_1 + x_2 = 6 \\ & x_1 + x_2 \leq 4 \\ & x_1 + 2x_2 \leq 5 \end{aligned}$$

with $x_1, x_2 \geq 0$.

6. Suppose we have obtained the tableau below for a maximization problem. State conditions on $a_1, a_2, a_3, b, c_1, c_2$ that are required to make the following statements true:

(a) The current solution is optimal, and there are alternative optimal solutions.

(b) The current basic solution is not a BFS.

(c) The current basic solution is a degenerate BFS.

(d) The current basic solution is feasible, but the LP is unbounded.

(e) The current basic solution is feasible, but the objective function can be improved by replacing x_6 with x_1 as a basic variable.

x_1	x_2	x_3	x_4	x_5	x_6	rhs
c_1	c_2	0	0	0	0	10
4	a_1	1	0	a_2	0	b
−1	−5	0	1	−1	0	2
a_3	−3	0	0	−4	1	3

7. (This is from our Group Work Handout for Chapter 6)

In the textbook's "Dakota Problem", we are making desks, tables and chairs, and we want to maximize profit given constraints on lumber, finishing and carpentry (resp).

For the primal, let x_1, x_2, x_3 be the number of desks, tables and chairs we make (resp), where the original (max) tableau is as given below:

x_1	x_2	x_3	s_1	s_2	s_3	rhs
−60	−30	−20	0	0	0	0
8	6	1	1	0	0	48
4	2	$\frac{3}{2}$	0	1	0	20
2	$\frac{3}{2}$	$\frac{1}{2}$	0	0	1	8

→

x_1	x_2	x_3	s_1	s_2	s_3	rhs
0	5	0	0	10	10	280
0	−2	0	1	2	−8	24
0	−2	1	0	2	−4	8
1	$\frac{5}{4}$	0	0	$-\frac{1}{2}$	$\frac{3}{2}$	2

(a) Write the down vectors/matrices that we typically use in our computations. Namely, $\mathbf{c}, \mathbf{c}_B, B$, and B^{-1}.

(b) Using our vector notation, if $\mathcal{B}$ gives the optimal basis, how do we compute the dual, $\mathbf{y} =$?

(c) Write down the dual (either as an initial tableau or in "normal form").

(d) Using the optimal Row 0 from the primal, write down the solution to the dual:

(e) In our "normal form", we have $A\mathbf{x} \leq \mathbf{b}$ for the primal and $A^T\mathbf{y} \geq \mathbf{c}$ for the dual. We will define two "slacks"[1]

- The "slack" for the primal, given $\mathbf{x}$: $\mathbf{b} - A\mathbf{x}$. Compute the current slack for the primal.
- The "slack" for the dual, given $\mathbf{y}$: $A^T\mathbf{y} - \mathbf{c}$. Compute the current slack for the dual. You can use your answer to (3) if necessary.

(f) What is the shadow price for each constraint?

(g) Write down the inequalit(ies) we need for Δ if we change the coefficient of x_2 from 30 to $30 + \Delta$, and we want the current basis to remain optimal.

(h) Write down the inequalit(ies) we need for Δ if we change the coefficient of x_3 from 20 to $20 + \Delta$, and we want the current basis to remain optimal.

(i) How does changing a *column* of A effect the dual? Use this to see what would happen if we change the column for x_2 (tables) to be $[5, 2, 2]^T$- Is it now worth it to make tables?

(j) How does creating a new column of A effect the dual? Use this to see if it makes sense to manufacture footstools, where we sell them for \$15 each, and the resources are $[1, 1, 1]^T$.

8. Consider the LP and the optimal tableau with missing Row 0 and missing optimal RHS (assume big-M).

$$\begin{aligned} \max z = \; & 3x_1 + x_2 \\ \text{s.t.} \; & 2x_1 + x_2 \leq 4 \\ & 3x_1 + 2x_2 \geq 6 \\ & 4x_1 + 2x_2 = 7 \\ & x_1, x_2 \geq 0 \end{aligned}$$

x_1	x_2	s_1	e_2	a_2	a_3	rhs
0	0	1	0	0	$-1/2$	
0	1	0	-2	2	$-3/2$	
1	0	0	1	-1	1	

Find Row 0 and the RHS for the optimal tableau (without performing row reductions!)

9. Give an argument why, if the primal is unbounded, then the dual must be infeasible.

10. Consider the following LP and its optimal tableau, shown.

$$\begin{aligned} \max z = \; & 4x_1 + x_2 \\ \text{st} \; & x_1 + 2x_2 = 6 \\ & x_1 - x_2 \geq 3 \\ & 2x_1 + x_2 \leq 10 \\ & x_1, x_2 \geq 0 \end{aligned} \quad \Rightarrow$$

x_1	x_2	e_1	s_1	a_1	a_2	*rhs*
0	0	0	$7/3$	$M - 2/3$	M	$58/3$
0	1	0	$-1/3$	$2/3$	0	$2/3$
1	0	0	$2/3$	$-1/3$	0	$14/3$
0	0	1	1	-1	-1	1

(a) Find the dual of this LP and its optimal solution.

(b) Find the range of values of $b_3 = 10$ for which the current basis remains optimal.

[1] Sorry, the vocabulary is related to the "slack variable", but we're using "slack" in a different context now. Ask if you're ever not sure which we're talking about.

11. Televco produces TV tubes at three plants, shown below. We have three customers, and the profits for each depend on the plant, as shown on the right.

Plant	1	2	3
Number of Tubes	50	100	50

	Cust 1	Cust 2	Cust 3
Plant 1	75	60	69
Plant 2	79	73	68
Plant 3	85	76	70

- Formulate a balanced transportation porblem that can be used to maximize profits.
- Use the NW corner method to find a BFS to the problem.
- Find an optimal solution.

12. Five workers are available to perform four jobs. The time it takes each worker to perform each job is given below. The goal is to assign workers to jobs so as to minimize the total time required. Use the Hungarian method to solve.

	Job 1	2	3	4
Worker 1	10	15	10	15
2	12	8	20	16
3	12	9	12	18
4	6	12	15	18
5	16	12	8	12

3. A company must meet the demands shown below for a product. Demand may be backlogged at a cost of \$5 per unit per month. All demand must be met at the end of March. Thus, if 1 unit of January demand is met during March, a cost of \$5× 2=\$10 is incurred. Monthly production capacity and unit production cost during each month are shown below. A holding cost of \$20 per unit is assessed on the inventory at the end of each month.

Month	Demand	Prod Cap	Unit Prod Cost
Jan	30	35	400
Feb	30	30	420
Mar	20	35	410

Formulate a balanced transportation problem that can be used to determine how to minimize the cost (including backlogging, holding and production costs).

HINT: To set this up, think of Jan, Feb and Mar as having supplies of $35, 30 and 35$, and demands of $30, 30, 20$ (we'll need a dummy to balance. For the costs, January can supply January at a cost of \$400 per unit, January can supply Feb at a cost of \$420 per unit, and it can supply March at a cost of \$440 per unit.

13. Solve the following LP (HINT: It can be put into a 2×2 transportation problem).

$$\begin{aligned} \min z = \quad & 2x_1 + 3x_2 + 4x_3 + 3x_4 \\ s.t. \quad & x_1 + x_2 \leq 4 \\ & x_3 + x_4 \leq 5 \\ & x_1 + x_3 \geq 3 \\ & x_2 + x_4 \geq 6 \end{aligned}$$

(All variables ≥ 0).

14. Find the optimal solution to the balanced transportation problem below:

[4]	[2]	[4]	15
[12]	[8]	[4]	15
10	10	10	

15. Consider the optimal tableau for the PowerCo problem:

[8]	[6] 10	[10] 25	[9]	35
[9] 45	[12]	[13] 5	[7]	50
[14]	[9] 10	[16]	[5] 30	40
45	20	30	30	

(a) Find the range of values of c_{24} for which the current basis is optimal.

(b) Find the range of values of c_{23} for which the current basis is optimal.

16. Write the shortest path problem as (i) a transhipment problem, and (ii) a linear program. For specificity, use the PowerCo network below (Figure 2, p 414). (Hints: For transhipment, we have one supply, one demand, and a bunch of warehouses. For the LP, you could write it from the transhipment problem.). Finally, find the shortest path from Plant 1 to City 1 using Dijkstra's algorithm.

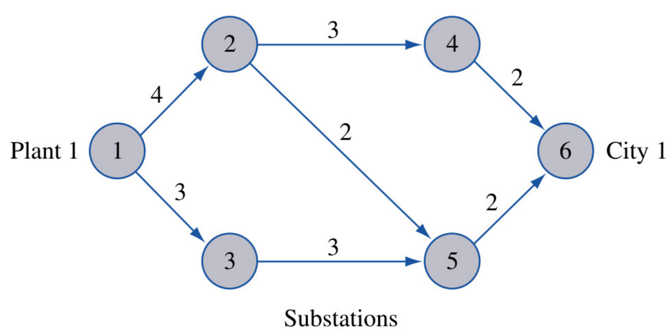
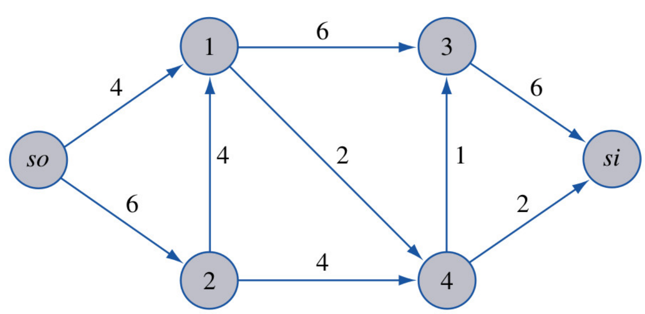


FIGURE 23

17. Given the figure below (Fig 23 from the text), first write the maximum flow problem as a linear program. (Hint: Think about the constraints on the flow for each edge, then for each vertex). Solve the max-flow problem using Ford-Fulkerson. Be sure to write out the residual graphs. Finally, find a cut giving the minimum capacity to show that your solution is correct.



18. Continuing with Figure 23 from the previous question, with the maximum flow, if the cut is:

$$A = \{so, 2, 3\}, B = \{1, 4, si\}$$

then what is the net flow across the cut? What is the capacity of the cut?

19. For each of the following projects, draw the network, then find the critical path by first computing ES, EF, LS, LF and the slack for each activity.

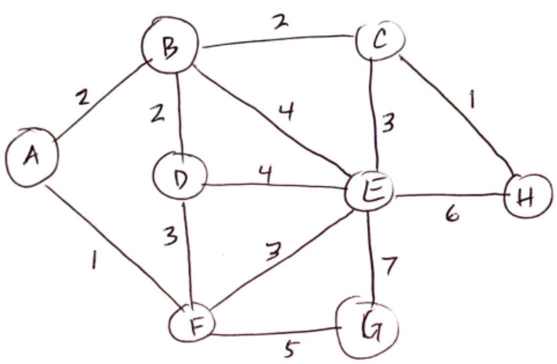
(a)

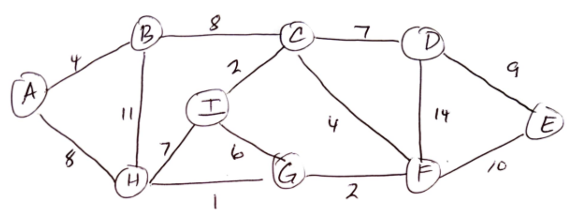
Activity	Precedents	Duration
A	–	15
B	–	20
C	A	25
D	A	10
E	B	15
F	B	20
G	D,E	20
H	D,E	30
I	D,E	15
J	C,G	10
K	F,I	20

(b)

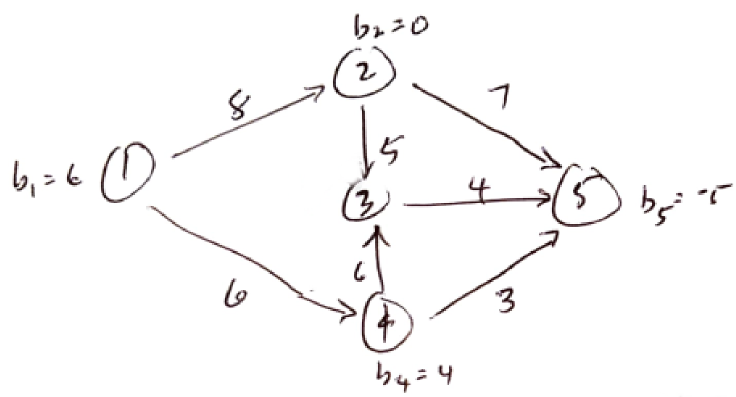
Activity	Precedents	Duration
A	–	7
B	–	9
C	A	12
D	A,B	8
E	D	9
F	C,E	6
G	E	5

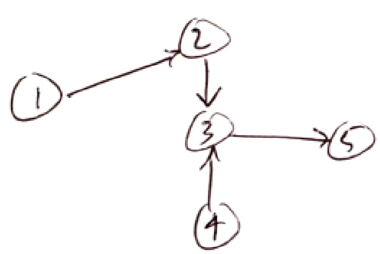
20. For the following graphs, find the minimum spanning tree.





21. Consider the diagram to the left with **costs** along the edges, and the diagram to the right which is a given spanning tree.



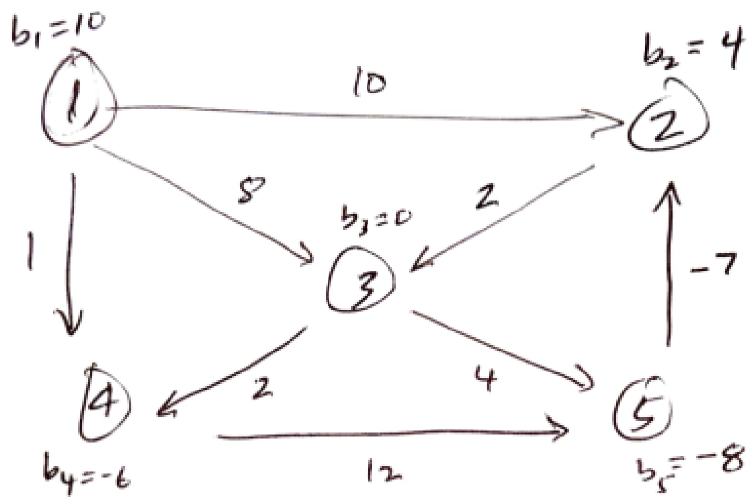


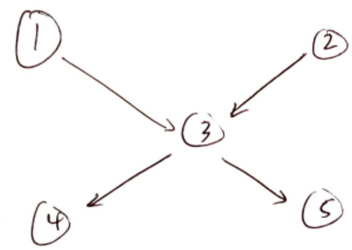
(a) What value should b_3 be for this to be a balanced problem (note that $b_5 = -5$; it might be hard to read). Assume that b_3 is this value for the rest of the problem below.

(b) Find the flow values for the spanning tree and determine if it represents a BFS to the minimization problem.

(c) Use the network simplex method to solve the minimization problem.

22. Consider the diagram to the left with **costs** along the edges, and the diagram to the right which is a given spanning tree.





(a) Find the flow values for the spanning tree and determine if it represents a BFS to the minimization problem.

(b) Use the network simplex method to solve the minimization problem.