Sensitivity Analysis

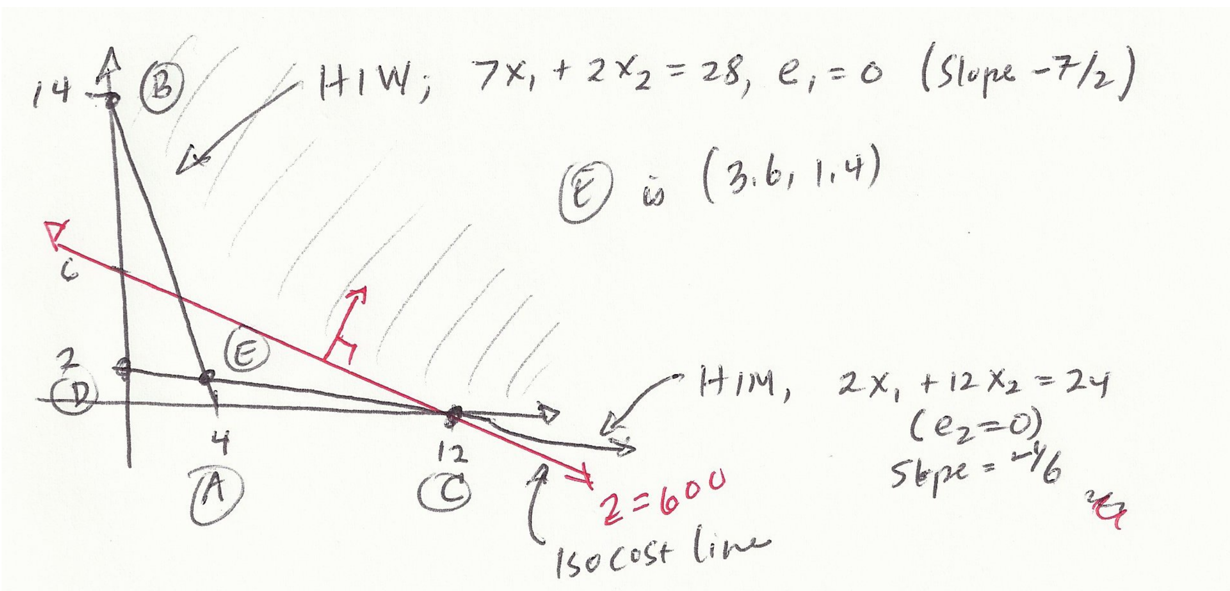
This is Example 2, Section 3.2: We're buying advertising time for our two target demos: "HIW" and "HIM" (High Income Women and High Income Men).

Let x_1, x_2 be the number of ads purchased during a comedy show, and a football game respectively. Comedy ads are \$50,000 and football ads are \$100,000. For the target demos, we want to solve the following LP:

$$\begin{aligned} \min z = \quad & 50x_1 + 100x_2 \\ \text{st} \quad & 7x_1 + 2x_2 \geq 28 \qquad \text{(HIW)} \\ & 2x_1 + 12x_2 \geq 24 \qquad \text{(HIM)} \\ & x_1, x_2 \geq 0 \end{aligned}$$

where the unit of money is \$1,000 and the unit of people is in millions.

Solving this graphically:



In this case, we see that the point $E = (3.6, 1.4)$ is the unique optimal solution.

Now we'll look at how sensitive the solution is to changes in the parameters:

1. Find the range of values of the cost of x_1 for which the current basis remains feasible (the basis is the set of basic variables).

   SOLUTION: Slope of the isocost line is between the other 2 slopes.

   $$k = c_1 x_1 + 100x_2 \quad \Rightarrow \quad x_2 = -\frac{c_1}{100}x_1 + \frac{k}{100}$$

   To have the current basis remain the (unique) optimal solution, we must have:

   $$-\frac{7}{2} < -\frac{c_1}{100} < \frac{-1}{6} \quad \Rightarrow \quad \frac{50}{3} < c_1 < 350$$

   If we allowed for multiple optimal solutions, we could put equalities there.

2. (For class) Find the range of values of the cost of a football ad for which the current basis remains optimal.

   SOLUTION: Again, slope of the isocost line is between the others. In this case,

$$-\frac{7}{2} < -\frac{50}{c_2} < -\frac{1}{6} \quad \Rightarrow \quad \frac{100}{7} < c_2 < 300$$

3. Find the range of exposures of required HIW for which the current basis (x_1, x_2) remains optimal.

   SOLUTION: Look at $7x_1 + 2x_2 = b_1$.

   - In changing b_1, the slope stays the same; the line drops (in a parallel way) or lifts.
   - Having the same basis means that the optimal value will be attained at the intersection of HIW and HIM.
   - As b_1 decreases, the point **E** slides toward point $\mathbf{D}(0, 2)$. This occurs when:

$$b_1 = 7x_1 + 2x_2 = 7(0) + 2(2) = 4$$

     Therefore, $b_1 > 4$.
   - Similarly, as b_1 increases, **E** slides toward $\mathbf{C}(12, 0)$, and they intersect when

$$b_1 = 7(12) + 2(0) = 84$$

   Therefore, the current basis remains optimal if

$$4 < b_1 < 84$$

   Sometimes it is more convenient to see these results in terms of the change to the current value $b_1 = 28 + \Delta$:

$$4 < 28 + \Delta < 84 \quad \Rightarrow \quad -24 < \Delta < 56$$

   We can also determine the new point of intersection if we want to move the constraint by Δ-

   We solve for the point of intersection of the system below using Cramer's Rule. Using the current basis, x_1 and x_2 are basic and e_1, e_2 are set to zero.

$$\begin{aligned} 7x_1 + 2x_2 &= 28 + \Delta \\ 2x_1 + 12x_2 &= 24 \end{aligned}$$

$$x_1 = \frac{\begin{vmatrix} 28+\Delta & 2 \\ 24 & 12 \end{vmatrix}}{\begin{vmatrix} 7 & 2 \\ 2 & 12 \end{vmatrix}} = \frac{336 + 12\Delta - 48}{7 \cdot 12 - 4} = \frac{18}{5} + \frac{3}{20}\Delta = 3.6 + 0.15\Delta$$

   And

$$x_2 = \frac{\begin{vmatrix} 7 & 28+\Delta \\ 2 & 24 \end{vmatrix}}{\begin{vmatrix} 7 & 2 \\ 2 & 12 \end{vmatrix}} = \frac{168 - 56 - 2\Delta}{7 \cdot 12 - 4} = \frac{7}{5} - \frac{1}{40}\Delta = 1.4 - 0.025\Delta$$

4. Repeat, but find the sensitivity to the HIM constraint.

   SOLUTION: As before, $2x_1 + 12x_2 = b_2$ (locate the bottom line on the graph). Therefore,

   - As b_2 increases, point **E** slides toward $\mathbf{B}(0, 14)$.

   $$2(0) + 12(14) = b_2 \quad \Rightarrow \quad b_2 = 168$$

   - As b_2 decreases, point **E** slides toward $\mathbf{A}(4, 0)$.

   $$2(4) + 12(0) = b_2 \quad \Rightarrow \quad b_2 = 8$$

   Therefore,

   $$8 \le b_2 \le 168$$

   So, in terms of Δ, we could compute the range directly:

   $$8 \le 24 + \Delta \le 168 \quad \Rightarrow \quad -16 \le \Delta \le 144$$

   And for the point of intersection between the constraints (note the basis)

   $$\begin{aligned} 7x_1 + 2x_2 &= 28 \\ 2x_1 + 12x_2 &= 24 + \Delta \end{aligned}$$

   $$x_1 = \frac{\begin{vmatrix} 28 & 2 \\ 24+\Delta & 12 \end{vmatrix}}{\begin{vmatrix} 7 & 2 \\ 2 & 12 \end{vmatrix}} = \frac{336 - 48 - 2\Delta}{80} = \frac{18}{5} - \frac{1}{40}\Delta = 3.6 - 0.025\Delta$$

   And

   $$x_2 = \frac{\begin{vmatrix} 7 & 28 \\ 2 & 24+\Delta \end{vmatrix}}{\begin{vmatrix} 7 & 2 \\ 2 & 12 \end{vmatrix}} = \frac{168 - 56 + 7\Delta}{80} = \frac{7}{5} + \frac{7}{80}\Delta = 1.4 + 0.0875\Delta$$

Shadow Prices, Part I

The shadow price (of a constraint) is the amount by which the value of the objective function *is improved* (increased in a maximum, decreased in a minimum) when the RHS of the constraint is *increased by 1 unit.* This is done *under the assumption that the current basis is unchanged.*

Continuing with our Example

Find the shadow price of each constraint.

SOLUTION:

- For HIW, if the RHS is $28 + \Delta$, then we solve the system below, which we already did:

$$\begin{aligned} 7x_1 + 2x_2 &= 28 + \Delta \\ 2x_1 + 12x_2 &= 24 \end{aligned} \quad \Rightarrow \quad x_1 = 3.6 + 0.15\Delta, x_2 = 1.4 - 0.025\Delta$$

Substitute these into the objective function:

$$(\min)z = 50(3.6 + 0.15\Delta) + 100(1.4 - 0.025\Delta) = 320 + 5\Delta$$

Increasing the RHS of constraint 1 will represent a *worse* value of z by 5 units. That means the shadow price is -5 (because the minimum is worse).

- Similarly for HIM, if the RHS is $24 + \Delta$, then we solve the system below:

$$\begin{aligned} 7x_1 + 2x_2 &= 28 \\ 2x_1 + 12x_2 &= 24 + \Delta \end{aligned} \quad \Rightarrow \quad x_1 = 3.6 - 0.025\Delta, x_2 = 1.4 + 0.0875\Delta$$

Substitute these into the objective function:

$$(\min)z = 50(3.6 - 0.025\Delta) + 100(1.4 + 0.0875\Delta) = 320 + 7.5\Delta$$

Increasing the RHS of constraint 2 by 1 unit will represent a *worse* value of z by 7.5 units. That means the shadow price is -7.5.

Continuing the example...

If 26 HIW exposures are required (and other parameters remain the same), determine the new solution and the new value of z.

SOLUTION: We've already done the computations- Just set $\Delta = -2$ (which is in the allowable range). Then the optimal solution changes slightly:

$$x_1 = 3.6 + 0.15(-2) = 3.30 \qquad x_2 = 1.4 - 0.025(-2) = 1.45$$

and $z = 320 + 5(-2) = 310$ (double-check: $50(3.3) + 100(1.45) = 310$)

LINGO

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min=50*x1+100*x2;
7*x1+2*x2>=28;
2*x1+12*x2>=24;
```

The solution report gives us the shadow prices (dual prices):

```
  Variable           Value        Reduced Cost
        X1        3.600000            0.000000
        X2        1.400000            0.000000
       Row    Slack or Surplus      Dual Price
         1        320.0000           -1.000000
         2        0.000000           -5.000000
         3        0.000000           -7.500000
```

And in the range report:

```
 RANGES IN WHICH THE BASIS IS UNCHANGED:

                           OBJ COEFFICIENT RANGES
 VARIABLE         CURRENT        ALLOWABLE        ALLOWABLE
                   COEF          INCREASE         DECREASE
       X1       50.000000       300.000000        33.333332
       X2      100.000000       200.000000        85.714287

                           RIGHTHAND SIDE RANGES
      ROW         CURRENT        ALLOWABLE        ALLOWABLE
                    RHS          INCREASE         DECREASE
        2       28.000000        56.000000        23.999998
        3       24.000000       144.000000        16.000000
```