Extra Worked Example: Full Sensitivity Analysis

A factory can produce 4 products. Each product must be processed in each of two workshops. The processing times and profit margins for each of the four products is shown.

	1	2	3	4
Workshop 1	3	4	8	6
Workshop 2	6	2	5	8
Profit	4	6	10	9

If we have 400 hours of labor available in each workshop, the following LP can be used:

$$
\begin{array}{llllll}
\max\ z = & 4x_1 & +6x_2 & +10x_3 & +9x_4 & \\
\text{st} & 3x_1 & +4x_2 & +8x_3 & +6x_4 & \leq 400 \text{ Labor 1} \\
 & 6x_1 & +2x_2 & +5x_3 & +8x_4 & \leq 400 \text{ Labor 2}
\end{array}
$$

The initial and final tableaux:

x_1	x_2	x_3	x_4	s_1	s_2	
-4	-6	-10	-9	0	0	0
3	4	8	6	1	0	400
6	2	5	8	0	1	400

x_1	x_2	x_3	x_4	s_1	s_2	
$1/2$	0	2	0	$3/2$	0	600
$3/4$	1	2	$3/2$	$1/4$	0	100
$9/2$	0	1	5	$-1/2$	1	200

Sensitivity Analysis

The basic variables are (in order): $\mathcal{B} = \{x_2, s_2\}$ so that the matrices B and B^{-1} can be read directly from the initial and final tableaux respectively.

$$
B = \begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 1/4 & 0 \\ -1/2 & 1 \end{bmatrix}
$$

Further, the vector $\mathbf{c}^T = [4, 6, 10, 9, 0, 0]$ and the vector $\mathbf{c}_B^T = [6, 0]$.

1. Sensitivity Analysis on the NBVs.

   - x_1: Change 4 to $4 + \Delta$.
     We see that $\hat{c}_1 = 1/2$, so we have $\hat{c}_k - \Delta > 0$: $\frac{1}{2} - \Delta > 0 \quad \Rightarrow \quad \Delta < \frac{1}{2}$.
   - x_3: Change 10 to $10 + \Delta$. Again, $\hat{c}_k - \Delta > 0$ gives: $2 - \Delta > 0 \quad \Rightarrow \quad \Delta < 2$.
   - x_4: Change 9 to $9 + \Delta$, and we have $-\Delta > 0$ or $\Delta < 0$.
   - We could also ask change the value of s_1 (in z). By the same reasoning of the previous variables, we would get $\frac{3}{2} - \Delta > 0$.

2. Sensitivity of BVs.

   - Change x_2 from 6 to $6 + \Delta$.
     Since x_2 is the first variable in $\mathbf{c}_B$, we use the first row of $B^{-1}A$ for our computation. Take the final Row 0 and add $\Delta\times$ Row 1. I have crossed out the BV positions, which sum to zero.

$$
\begin{array}{rcccccc}
 & 1/2 & X & 2 & 0 & 3/2 & X \\
+ & 3\Delta/4 & X & 2\Delta & 3\Delta/2 & \Delta/4 & X \\
\hline
 & \frac{1}{2} + \frac{3}{4}\Delta & 0 & 2 + 2\Delta & \frac{3}{2}\Delta & \frac{3}{2} + \frac{1}{4}\Delta & 0
\end{array}
$$

   We want all four non-zero expressions to be non-negative. Take the intersection of the four intervals, and we should see in this case that $\Delta > 0$ will satisfy all four.

- Change s_2 from 0 to Δ.
  This is the second item in $\mathbf{c}_B$, so use the second row in the final tableau:

$$\begin{array}{rcccccc} & 1/2 & X & 2 & 0 & 3/2 & X \\ + & 9\Delta/2 & X & \Delta & 5\Delta & -\Delta/2 & X \\ \hline & \frac{1}{2}+\frac{9}{2}\Delta & 0 & 2+\Delta & 5\Delta & \frac{3}{2}-\frac{1}{2}\Delta & 0 \end{array} \quad \Rightarrow \quad 0 < \Delta < 3$$

3. Changes in the RHS and the Shadow Prices.

   - Change in the first constraint:
     Add the RHS of the final tableau to $\Delta\times$ the first column of B^{-1}.

$$\begin{bmatrix} 100 \\ 200 \end{bmatrix} + \Delta \begin{bmatrix} 1/4 \\ -1/2 \end{bmatrix} \quad \Rightarrow \quad z = 6(100 + \Delta/4) = 600 + \frac{3}{2}\Delta$$

     The shadow price for the first constraint is 3/2.

   - Change in the second constraint.
     Add the RHS of the final tableau to $\Delta\times$ the second column of B^{-1}:

$$B^{-1}\mathbf{b} + \Delta B_2^{-1} = \begin{bmatrix} 100 \\ 200 \end{bmatrix} + \Delta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

     so that $z = 600$. The shadow price is 0.

     NOTE: It makes sense that the shadow price is zero- In the optimal tableau, if $s_2 = 200$, then we have an extra 200 hours of labor available. Increasing that by 1 does nothing to z.

4. What if we introduce a new product, x_5, that has a profit of \$12.00 per unit, but is a process hog: $[8, 8]^T$. Would it be worth it to bring this product in?

   SOLUTION: To price out the column, treat it as a new column of A. The new column in $B^{-1}A$ is then

$$B^{-1}\mathbf{a}_5 = \begin{bmatrix} 1/4 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

   And the new Row 0 element would be

$$-12 + [6, 0] \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

   This means that bringing in Product 3 would actually create multiple optimal solutions, but would not increase the overall profit z.