Example

$$\begin{array}{lll} \max & 6x_1 & +5x_2 \\ \text{s.t.} & x_1 & +x_2 \leq 5 \\ & 3x_1 & +2x_2 \leq 12 \end{array}$$

with $x_1, x_2 \geq 0$. Find an upper bound to the maximum.

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How about a linear combination?

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$$y_1(x_1 \quad +x_2) \quad \leq 5y_1$$

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$$\min \quad 5y_1 + 12y_2$$

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Generalized, we have a "normal" max problem:

$$\begin{array}{rl} \max z = & \mathbf{c}^T\mathbf{x} \\ \text{st} & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{array} \quad \Longleftrightarrow$$

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Note: **b** may have negative values in this construction.

$$\begin{aligned} \max z = \quad & \mathbf{c}^T\mathbf{x} \\ \text{st} \quad & A\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge 0 \end{aligned} \qquad \iff \qquad \begin{aligned} \min w = \quad & \mathbf{b}^T\mathbf{y} \\ \text{st} \quad & A^T\mathbf{y} \ge \mathbf{c} \\ & \mathbf{y} \ge 0 \end{aligned}$$

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Same as number of variables in primal.

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- The RHS of primal $\rightarrow$ obj function coeffs in dual.

Self test: Create the Dual

$$
\begin{array}{llll}
\max & 2x_1 & +3x_2 & -x_3 \\
\text{s.t.} & x_1 & +2x_2 & +x_3 \leq 1 \\
 & x_1 & -x_2 & -x_3 \leq 5
\end{array}
\qquad \Longleftrightarrow
$$

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An example with non-"normal" issues. Find the dual for:

$$\begin{aligned} \min \quad & 8x_1 + 5x_2 + 4x_3 \\ \text{s.t.} \quad & 4x_1 + 2x_2 + 8x_3 = 12 \\ & 7x_1 + 5x_2 + 6x_3 \geq 9 \\ & 8x_1 + 5x_2 + 4x_3 \leq 10 \\ & 3x_1 + 7x_2 + 9x_3 \geq 7 \\ & x_1 \geq 0, x_2 \text{ URS}, x_3 \leq 0 \end{aligned}$$

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We'll put this system into "normal form":

- Change the min to a max.
- Multiply Constraints 2 and 4 by -1.
- Change $=$ into inequalities.
- Find substitutions for x_2, x_3.

First three issues:

$$
\begin{array}{lrrrl}
\max & -8x_1 & -5x_2 & -4x_3 & \\
\text{s.t.} & 4x_1 & +2x_2 & +8x_3 & \leq 12 \\
 & -4x_1 & -2x_2 & -8x_3 & \leq -12 \\
 & -7x_1 & -5x_2 & -6x_3 & \leq -9 \\
 & 8x_1 & +5x_2 & +4x_3 & \leq 10 \\
 & -3x_1 & -7x_2 & -9x_3 & \leq -7
\end{array}
$$

$$x_1 \geq 0, x_2 \text{ URS}, x_3 \leq 0$$

Last issue:

First three issues:

$$\begin{array}{lrrrl} \max & -8x_1 & -5x_2 & -4x_3 & \\ \text{s.t.} & 4x_1 & +2x_2 & +8x_3 & \leq 12 \\ & -4x_1 & -2x_2 & -8x_3 & \leq -12 \\ & -7x_1 & -5x_2 & -6x_3 & \leq -9 \\ & 8x_1 & +5x_2 & +4x_3 & \leq 10 \\ & -3x_1 & -7x_2 & -9x_3 & \leq -7 \end{array}$$

$$x_1 \geq 0, x_2 \text{ URS}, x_3 \leq 0$$

Last issue: Let $x_2 = x_4 - x_5$ and $x_3 = -x_6$ (with all these new vars non-neg)

Now a summary of the equations using a table:

$x_1 \geq 0$	$x_4 \geq 0$	$x_5 \geq 0$	$x_6 \geq 0$	
4	2	-2	-8	≤ 12
-4	-2	2	8	≤ -12
-7	-5	5	6	≤ -9
8	5	-5	-4	≤ -12
-3	-7	7	9	≤ -7
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This is standard form. This table is given in the form:

$\mathbf{x}$	
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$\Rightarrow$

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$$\Rightarrow \qquad \begin{aligned} \min w = \quad & \mathbf{b}^T\mathbf{p} \\ \text{st} \quad & A^T\mathbf{p} \geq \mathbf{c} \\ & \mathbf{p} \geq 0 \end{aligned}$$

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Note that $\mathbf{p}$ has 5 variables (from the 5 constraints).

Let $p_1, \ldots, p_5$ be the new vars;

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$$\begin{aligned}
\min w = \quad & 12p_1 - 12p_2 - 9p_3 + 10p_4 - 7p_5 \\
& 4p_1 - 4p_2 - 7p_3 + 8p_4 - 3p_5 \geq -8 \\
& 2p_1 - 2p_2 - 5p_3 + 5p_4 - 7p_5 \geq -5 \\
& -2p_1 + 2p_2 + 5p_3 - 5p_4 + 7p_5 \geq 5 \\
& -8p_1 + 8p_2 + 6p_3 - 4p_4 + 9p_5 \geq 4 \\
& p_1, p_2, p_3, p_4, p_5 \geq 0
\end{aligned}$$

In the next frame, we'll start simplifying this system. Look to see what we might do first.

- Variables p_1, p_2 can be combined: $p_6 = p_1 - p_2$ (now is URS).
- Constraints 2 and 3 can be combined into equality.
- Multiply first constraint by -1 to get positive b (next page).

$$
\begin{aligned}
\min w = \quad & 12(p_1 - p_2) - 9p_3 + 10p_4 - 7p_5 \\
& 4(p_1 - p_2) - 7p_3 + 8p_4 - 3p_5 \geq -8 \\
& 2(p_1 - p_2) - 5p_3 + 5p_4 - 7p_5 \geq -5 \\
& -2(p_1 - p_2) + 5p_3 - 5p_4 + 7p_5 \geq 5 \\
& -8(p_1 - p_2) + 6p_3 - 4p_4 + 9p_5 \geq 4 \\
& p_1, p_2, p_3, p_4, p_5 \geq 0
\end{aligned}
$$

$$\begin{array}{lrrrrl} \min w = & 12p_6 & -9p_3 & +10p_4 & -7p_5 & \\ & -4p_6 & +7p_3 & -8p_4 & +3p_5 & \leq 8 \\ & -2p_6 & +5p_3 & -5p_4 & +7p_5 & = 5 \\ & -8p_6 & +6p_3 & -4p_4 & +9p_5 & \geq 4 \end{array}$$

with p_6 URS, and $p_3, p_4, p_5 \geq 0$.

- Let $p_7 = -p_6$ and $p_8 = -p_4$
- p_7 is URS, $p_8 \leq 0$, $p_3, p_5 \geq 0$.

Changing notation slightly:

$$\begin{aligned} \max z = \quad & 12x_1 + 9x_2 + 10x_3 + 7x_4 \\ \text{s.t.} \quad & 4x_1 + 7x_2 + 8x_3 + 3x_4 \leq 8 \\ & 2x_1 + 5x_2 + 5x_3 + 7x_4 = 5 \\ & 8x_1 + 6x_2 + 4x_3 + 9x_4 \geq 4 \\ & x_1 \text{ URS}, x_2 \geq 0, x_3 \leq 0, x_4 \geq 0 \end{aligned}$$

$$\begin{aligned} \min w = \quad & 8y_1 + 5y_2 + 4y_3 \\ \text{s.t.} \quad & 4y_1 + 2y_2 + 8y_3 = 12 \\ & 7y_1 + 5y_2 + 6y_3 \geq 9 \\ & 8y_1 + 5y_2 + 4y_3 \leq 10 \\ & 3y_1 + 7y_2 + 9y_3 \geq 7 \\ & y_1 \geq 0, y_2 \text{ URS}, y_3 \leq 0 \end{aligned}$$

Summary:

Primal:	Dual:
max	min

Summary:

Primal:	Dual:	
max	min	normal
$\leq$ constraint	≥ 0 variable	

Summary:

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$\geq$ constraint		

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≥ 0 variable	$\geq$ constraint	normal
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Summary:

Primal:	Dual:	
max	min	normal
$\leq$ constraint	≥ 0 variable	normal
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Equality constraint	URS variable	
≥ 0 variable	$\geq$ constraint	normal
≤ 0 variable	$\leq$ constraint	
URS variable		

Summary:

Primal:	Dual:	
max	min	normal
$\leq$ constraint	≥ 0 variable	normal
$\geq$ constraint	≤ 0 variable	
Equality constraint	URS variable	
≥ 0 variable	$\geq$ constraint	normal
≤ 0 variable	$\leq$ constraint	
URS variable	Equality constraint	

Primal-Dual Table

$$\begin{aligned} \max z = \ & \mathbf{c}^T\mathbf{x} \\ \text{st} \ & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned} \iff \begin{aligned} \min w = \ & \mathbf{b}^T\mathbf{y} \\ \text{st} \ & A^T\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq 0 \end{aligned}$$

Main concerns:
If the inequalities in the primal are not normal, what happens in the dual?
If the variables are not normal in the primal, what happens in the dual?
(and vice-versa)

	x_1?	x_2?	$\cdots$	x_n?	
y_1?					b_1?
y_2?		A			b_2?
$\vdots$					$\vdots$
y_m?					b_m?
	c_1?	c_2?	$\cdots$	c_n?	

Using a table

Starting problem:

$$\begin{aligned}\min w = \quad & 8y_1 + 5y_2 + 4y_3 \\ \text{s.t.} \quad & 4y_1 + 2y_2 + 8y_3 = 12 \\ & 7y_1 + 5y_2 + 6y_3 \geq 9 \quad \text{with } y_1 \geq 0, y_2 \text{ URS}, y_3 \leq 0 \\ & 8y_1 + 5y_2 + 4y_3 \leq 10 \\ & 3y_1 + 7y_2 + 9y_3 \geq 7\end{aligned}$$

(Asterisks mark things that are different than “normal”)

	x_1?	x_2?	x_3?	x_4?	
$y_1 \geq 0$	4	7	8	3	?8
y_2 urs($*$)	2	5	5	7	?5
$y_3 \leq 0(*)$	8	6	4	9	?4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

Using a table

	x_1?	x_2?	x_3?	x_4?	
$y_1 \geq 0$	4	7	8	3	≤ 8
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	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

Using a table

	x_1?	x_2?	x_3?	x_4?	
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y_2 urs$(*)$	2	5	5	7	

Using a table

	x_1?	x_2?	x_3?	x_4?	
$y_1 \geq 0$	4	7	8	3	≤ 8
y_2 urs$(*)$	2	5	5	7	$= 5$
$y_3 \leq 0(*)$	8	6	4	9	?4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

Using a table

	x_1?	x_2?	x_3?	x_4?	
$y_1 \geq 0$	4	7	8	3	≤ 8
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Using a table

	x_1 URS	$x_2 \geq 0$	$x_3 \leq 0$	x_4?	
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Using a table

	x_1 URS	$x_2 \geq 0$	$x_3 \leq 0$	$x_4 \geq 0$	
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$y_3 \leq 0(*)$	8	6	4	9	≥ 4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

Using a table- Here are Primal and Dual

	x_1 URS	$x_2 \geq 0$	$x_3 \leq 0$	$x_4 \geq 0$	
$y_1 \geq 0$	4	7	8	3	≤ 8
y_2 urs$(*)$	2	5	5	7	$= 5$
$y_3 \leq 0(*)$	8	6	4	9	≥ 4
	$= 12(*)$	≥ 9	$\leq 10(*)$	≥ 7	

$$\max z = 12x_1 + 9x_2 + 10x_3 + 7x_4$$

$$\text{s.t.} \quad \begin{aligned} 4x_1 + 7x_2 + 8x_3 + 3x_4 &\leq 8 \\ 2x_1 + 5x_2 + 5x_3 + 7x_4 &= 5 \\ 8x_1 + 6x_2 + 4x_3 + 9x_4 &\geq 4 \end{aligned}$$

$$x_1 \text{ URS}, x_2 \geq 0, x_3 \leq 0, x_4 \geq 0$$

$$\min w = 8y_1 + 5y_2 + 4y_3$$

$$\text{s.t.} \quad \begin{aligned} 4y_1 + 2y_2 + 8y_3 &= 12 \\ 7y_1 + 5y_2 + 6y_3 &\geq 9 \\ 8y_1 + 5y_2 + 4y_3 &\leq 10 \\ 3y_1 + 7y_2 + 9y_3 &\geq 7 \end{aligned}$$

$$y_1 \geq 0, y_2 \text{ URS}, y_3 \leq 0$$

Example 2

Use a table to write the dual:

$$\begin{aligned} \max z = \quad & 2x_1 + x_2 \\ \text{st} \quad & x_1 + x_2 = 2 \\ & 2x_1 - x_2 \geq 3 \\ & x_1 - x_2 \leq 1 \\ & x_1 \geq 0, x_2 \text{ urs} \end{aligned}$$

Example 2

Use a table to write the dual:

$$\begin{aligned} \max z = \quad & 2x_1 + x_2 \\ \text{st} \quad & x_1 + x_2 = 2 \\ & 2x_1 - x_2 \geq 3 \\ & x_1 - x_2 \leq 1 \\ & x_1 \geq 0, x_2 \text{ urs} \end{aligned}$$

	$x_1 \geq 0$	x_2 urs$(*)$	
y_1 ?	1	1	$= 2(*)$
y_2 ?	2	-1	$\geq 3(*)$
y_3 ?	1	-1	≤ 1
	? 2	? 1	

Example 2

Use a table to write the dual:

$$\begin{aligned} \max z = \quad & 2x_1 + x_2 \\ \text{st} \quad & x_1 + x_2 = 2 \\ & 2x_1 - x_2 \geq 3 \\ & x_1 - x_2 \leq 1 \\ & x_1 \geq 0, x_2 \text{ urs} \end{aligned}$$

	$x_1 \geq 0$	x_2 urs(∗)	
y_1 ?	1	1	$= 2(*)$
y_2 ?	2	-1	$\geq 3(*)$
y_3 ?	1	-1	≤ 1
	? 2	? 1	

	$x_1 \geq 0$	x_2 urs(∗)	
y_1 urs	1	1	$= 2(*)$
$y_2 \leq 0$	2	-1	$\geq 3(*)$
$y_3 \geq 0$	1	-1	≤ 1
	≥ 2	$= 1$	

Use a table to write the dual:

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y_1 urs	1	1	$= 2(*)$
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	$x_1 \geq 0$	x_2 urs$(*)$	
y_1 urs	1	1	$= 2(*)$
$y_2 \leq 0$	2	-1	$\geq 3(*)$
$y_3 \geq 0$	1	-1	≤ 1
	≥ 2	$= 1$	

$$\begin{aligned} \min w = \quad & 2y_1 + 3y_2 + y_3 \\ \text{st} \quad & y_1 + 2y_2 + y_3 \geq 2 \\ & y_1 - y_2 - y_3 = 1 \\ & y_1 \text{ urs}, y_2 \leq 0, y_3 \geq 0 \end{aligned}$$

Last Example

Find the dual associated with the following primal:

$$
\begin{aligned}
\max \quad & z = 3x_1 + x_2 \\
\text{st} \quad & 2x_1 + x_2 \leq 4 \\
& 3x_1 + 2x_2 \geq 6 \\
& 4x_1 + 2x_2 = 7 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Solution:

	$x_1 \geq 0$	$x_2 \geq 0$	
$y_1 \geq 0$	2	1	≤ 4
$y_2 \leq 0$	3	2	$\geq 6(*)$
y_3 URS	4	2	$= 7(*)$
	≥ 3	≥ 1	

$$\begin{aligned} \max \quad & z = 4y_1 + 6y_2 + 7y_3 \\ \text{st} \quad & 2y_1 + 3y_2 + 4y_3 \geq 3 \\ & y_1 + 2y_2 + 2y_3 \geq 1 \\ & y_1 \geq 0, y_2 \leq 0, y_3 \text{ URS} \end{aligned}$$