8.5: MCNFP

In this section, we take a look at the "Minimum Cost Network Flow Problem", or MCNFP. We will see that the general structure of this network allows us to write many of our network problems this way. We start with an example, and let's define some terms that will help us with conversions.
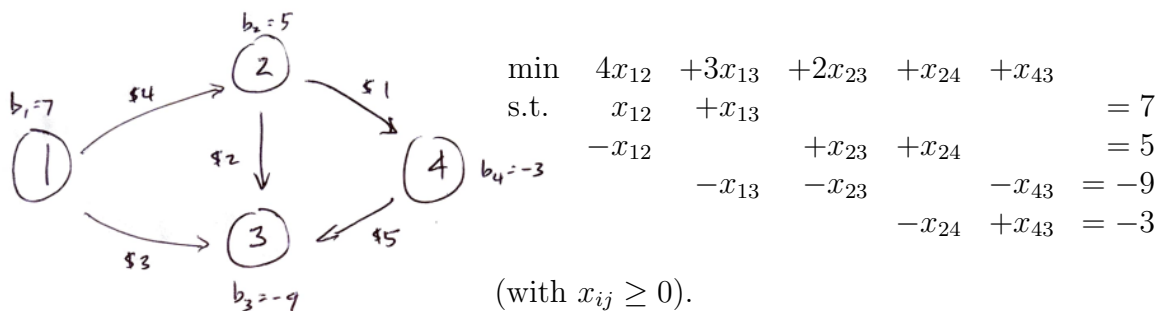
- In the LP, the arc from node i to node j will have an associated value, x_{ij} representing the flow on the edge.

- The arc from node i to node j will have a **cost**, c_{ij}, and each node i will have a **supply/demand** value, $b(i)$, where $b(i) > 0$ for a supply node, and $b(i) < 0$ for a demand node (and $b(i) = 0$ for a transhipment node).

- The LP is formed using the conservation property at each node. That is, either "flow in=flow out", or $Output - Input = b(i)$

Below, we'll take a simple example and write out the associated LP. Following that, we'll look back at the transportation tableau and the maximum flow problem.

Example 1

In the network below, notice that the problem is balanced, and the values on each edge are costs (rather than capacities). You might note that there is:

- one decision variable x_{ij} for each edge,

- one conservation constraint for each node ("outflow" $-$ "inflow" $= b(i)$)

$$\begin{array}{lrrrrrr}
\min & 4x_{12} & +3x_{13} & +2x_{23} & +x_{24} & +x_{43} & \\
\text{s.t.} & x_{12} & +x_{13} & & & & = 7 \\
& -x_{12} & & +x_{23} & +x_{24} & & = 5 \\
& & -x_{13} & -x_{23} & & -x_{43} & = -9 \\
& & & & -x_{24} & +x_{43} & = -3
\end{array}$$

(with $x_{ij} \geq 0$).

For future reference, you might notice the pattern of $+/-$ signs in each column. That's because each edge must begin at one node and end at another node- For example, edge $(1, 2)$ starts at node 1, so in constraint 1 we see x_{12}, then ends on node 2, so we see $-x_{12}$ in constraint 2.

The Minimum Cost Network Flow Problem (MCNFP)

The only addition to the variables we've already seen would be to add an upper and lower constraint for the flow (for example, the capacity of an edge would be an upper constraint). This would add a constraint of the type:

$$L_{ij} \leq x_{ij} \leq U_{ij} \text{ for all } i, j$$

Now the general LP can be written as

$$
\begin{aligned}
\min \quad & \sum \sum c_{ij} x_{ij} \\
\text{st} \quad & \sum_j x_{ij} - \sum_k x_{ki} = b(i) \qquad \text{for node } i = 1, 2, \cdots \\
& L_{ij} \leq x_{ij} \leq U_{ij}
\end{aligned}
$$

Note that the problem is not feasible until the demands are balanced: $\sum_i b(i) = 0$

The Transportation Problem

Below, we have a transportation tableau set up between two supply points and two demand points (and the problem is balanced). We already know how to turn the tableau into a network (and vice versa). We also write the associated MCNFP underneath of that.
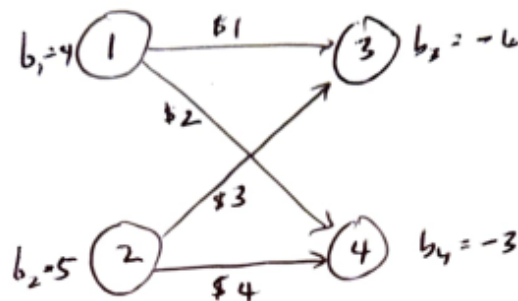

$$
\begin{array}{rcccc}
\min & x_{13} & +2x_{14} & +3x_{23} & +4x_{24} \\
& x_{13} & +x_{14} & & & = 4 \\
& & & x_{23} & +x_{24} & = 5 \\
& -x_{13} & & -x_{23} & & = -6 \\
& & -x_{14} & & -x_{24} & = -3
\end{array}
$$

The Assignment Problem

Much like the transportation problem, given an assignment tableau as shown below, we would create one node for each "person", and one node for each "job". Further, set $b(i) = +1$ for all "people" and $b(i) = -1$ for each "job" as shown.
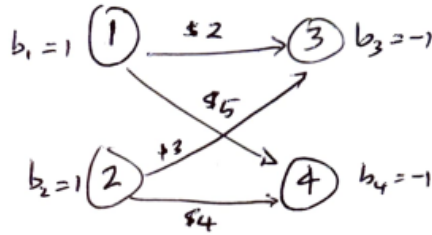
$$\min \quad 2x_{13} \quad +5x_{14} \quad +3x_{23} \quad +4x_{24}$$
$$x_{13} \quad +x_{14} \qquad\qquad\qquad = 1$$
$$x_{23} \quad +x_{24} \quad = 1$$
$$-x_{13} \qquad\qquad -x_{23} \qquad\qquad = -1$$
$$-x_{14} \qquad\qquad -x_{24} \quad = -1$$

The Shortest Path

The modifications for to convert a shortest path problem to MCNFP:

1. The costs are the values given on each edge.

2. First node is supply node, $b(1) = 1$; terminal node $b(4) = -1$. All others $b(i) = 0$.
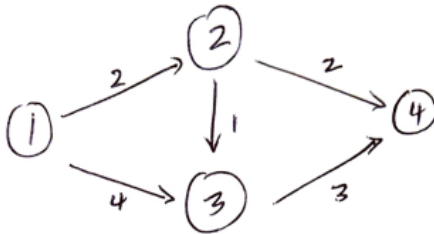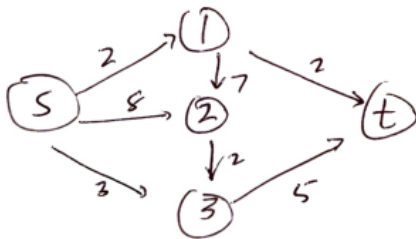

$$\min \quad 2x_{12} \quad +4x_{13} \quad +x_{23} \quad +2x_{24} \quad +3x_{34}$$
$$\text{st} \quad x_{12} \quad +x_{13} \qquad\qquad\qquad\qquad = 1$$
$$-x_{12} \qquad\qquad +x_{23} \quad +x_{24} \qquad\qquad = 0$$
$$-x_{13} \quad -x_{23} \qquad\qquad +x_{34} \quad = 0$$
$$-x_{24} \quad -x_{34} \quad = -1$$

(We could set $0 \le x_{ij} \le 1$).

The Maximum Flow

We let v be the value of the flow, and make $b(s) = v$, $b(t) = -v$ like we did in the shortest path. In the case of the flow, we also have capacities to worry about, so the upper bounds for x_{ij} are necessary in this case (with lower bounds typically set to 0).

$$\max \quad z = v$$
$$\text{st} \quad \sum_j x_{sj} - \sum_i x_{is} = v$$
$$\sum_j x_{ij} - \sum_i x_{ij} = 0 \quad i, j \ne s, t$$
$$\sum_j x_{tj} - \sum_i x_{it} = -v$$
$$L_{ij} \le x_{ij} \le U_{ij}$$

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Finally, we could set up the most general case- The case with costs on each edge, capacities (upper and lower) on each edge, then also supply and demand on each node. As we've seen, this general construction is very flexible and generalizes all the networks we've seen so far.

Homework

For each problem below, recast as a MCNFP:

1. A Transportation Problem: PowerCo, Table 39 in Section 7.4 (ignore the BFS and u, v values).

		City 1	City 2	City 3	City 4	Supply
	$v_j =$	6	6	10	2	
Plant 1	$u_i = 0$	8	6 10	10 25	9	35
Plant 2	3	9 45	12	13 5	7	50
Plant 3	3	14	9 10	16	5 30	40
Demand		45	20	30	30	

2. An Assignment Problem: Table 51 (Problem 2), Section 7.5

	Time (seconds)			
Swimmer	Free	Breast	Fly	Back
Gary Hall	54	54	51	53
Mark Spitz	51	57	52	52
Jim Montgomery	50	53	54	56
Chet Jastremski	56	54	55	53

3. A Shortest Path: Fig 2 in Section 8.2 (PowerCo again)

Substations
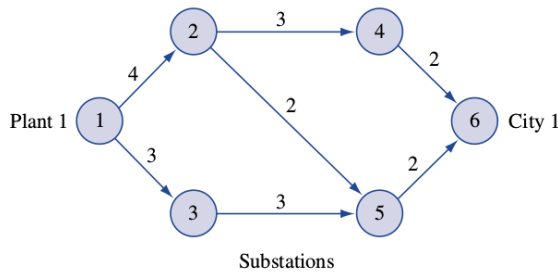
4. Max Flow: Figure 18, problem set for 8.3

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