Exam 1 Notes
Math 350
Spring 2025

You can prepare one half-page (8.5 x 6.5 or equivalent, single-sided) of notes to bring to the exam with you. I won't ask you directly about any programming languages, but there will be some coding in the take-home portion of the exam.

The exam will be 70% in class, and 30% take home (your choice of Matlab or Python). You are expected to do your own work, we'll talk about the take-home exam later.

Topic List

1. Some Basic Stats

   Know how to compute the mean, variance, and correlation by hand. Be able to double-center a matrix (one way is fine, you don't need to know all the alternatives). Know the definition of the covariance matrix. Be able to prove things about the variance and correlation (like you did in the homework for that section).

2. Linear Regression: Be able to find the line of best fit using the normal equations. Be able to set up linear regression problems for other linear models (for example, if the model equation was $y = c_1 \cos(x) + c_2 \sin(2x)$ instead of $y = mx + b$).

3. Basic linear algebra: Be able to define *coordinates* with respect to a particular basis. Given a basis, be able to compute the coordinates for a given vector. Know what it means to say that two vector spaces are "isomorphic". Be able to talk about the difference between the "high dimensional representation" of a given vector versus its "low dimensional representation".

   Define an "orthogonal" matrix. Be able to compute the projection of a vector onto another vector, and onto a subspace.

   Be able to compute a projection matrix for: a projection to a vector, a projection to a subspace spanned by orthonormal columns in a matrix, and projection to the column space of a general matrix A.

   Know the four fundamental subspaces associated to a matrix A (be able to draw a diagram like we did in class). Given the rank of A, give the dimensions of all four subspaces. Given a matrix and its RREF, be able to construct a basis for the column space, null space and row space.

   Show that the null space is orthogonal to the row space. Prove that if Q has orthonormal columns, then $\|Q\mathbf{x}\| = \|\mathbf{x}\|$, and that the dot product between Qx and Qy is the same as the dot product between x and y.

   Prove the Pythagorean Theorem on two vectors.

4. Eigenvalues/eigenvectors: Recall the three main equations used in eigenvalue/eigenvector computations and proofs. Be able to compute eigenvalues and eigenvectors for various matrices. Know the definition of an eigenspace (E_λ), and understand why it is a subspace (therefore, "solving for an eigenvector" is not well defined). Be able to prove some basic facts about eigenvalues. Is there a relationship between eigenvalues and invertibility?

5. The Spectral Theorem: We don't need all of the details here, mainly the following:

   "If and $n \times n$ matrix is symmetric, then it has n real eigenvalues and the set of eigenvectors can be formed so that they form an orthonormal basis for $\mathbb{R}^n$." Additionally, understand the relationship between the rank and the number of non-zero eigenvalues.

6. The SVD: Be able to state the Singular Value Decomposition, and define "singular values". Be able to compute the SVD by hand for "simple" matrices. Be able to use Matlab to compute the SVD, and from that, be able to find a basis for all four fundamental subspaces. From that, be able to project vectors to any of the four subspaces.

7. The Best Basis: Be able to compute the variance of a set of vectors projected to a single vector, and what relationship this has with the covariance matrix. Find the best basis for a given set of data (using Matlab), and be able to plot the best two dimensional representations.

Review Questions

1. Give mathematical formulas for the sample mean and sample variance. Give the formulas for (sample) covariance and correlation.

2. What is the definition of the covariance matrix for matrix X (say that X has p vectors in $\mathbb{R}^n$, and X is $n \times p$). You can define it by saying what the (i, j)th term of the covariance matrix represents.

3. Find the orthogonal projection of the vector $\boldsymbol{x} = [1, 0, 2]^T$ to the plane defined by:

$$G = \left\{ \alpha_1 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \text{ such that } \alpha_1, \alpha_2 \in \mathbb{R} \right\}$$

   Determine the distance from $\boldsymbol{x}$ to the plane G.

4. If $[\boldsymbol{x}]_{\mathcal{B}} = (3, -1)^T$, and $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$, what was $\boldsymbol{x}$ (in the standard basis)?

5. If $\boldsymbol{x} = (3, -1)^T$, and $\mathcal{B} = \left\{ \begin{bmatrix} 6 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$, what is $[\boldsymbol{x}]_{\mathcal{B}}$?

6. Let $\boldsymbol{a} = [1, 3]^T$. Find a square matrix A so that $A\boldsymbol{x}$ is the orthogonal projection of $\boldsymbol{x}$ onto the span of $\boldsymbol{a}$.

7. Determine the projection matrix that takes a vector $\mathbf{x}$ and outputs the projection of $\mathbf{x}$ onto the plane whose normal vector is $[1, 1, 1]^T$.

8. Find (by hand) the eigenvectors and eigenvalues of the matrix A:

$$A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

9. (Referring to the previous exercise) We could've predicted that the eigenvalues of the second matrix would be real, and that the eigenvectors would be orthogonal. Why?

10. Compute the SVD of the matrix A, and the pseudoinverse of A, given the matrix below:

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}$$

11. Compute the orthogonal projector to the span of $\mathbf{x}$, if $\mathbf{x} = [1, 1, 1]^T$.

12. Let

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

Find $[\mathbf{x}]_U$. Find the projection of $\mathbf{x}$ into the subspace spanned by the columns of U. Find the distance between $\mathbf{x}$ and the subspace spanned by the columns of U.

13. Show that $\text{Null}(A) \perp \text{Row}(A)$.

14. Show that, if X is invertible, then $X^{-1}AX$ and A have the same eigenvalues.

15. How do we "double-center" a matrix of data?

16. True or False, and give a short reason:

    (a) If the rank of A is 3, the dimension of the row space is 3.

    (b) If the correlation coefficient between two sets of data is 1, then the data sets are the same.

    (c) If the correlation coefficient between two sets of data is 0, then there is no functional relationship between the two sets of data.

(d) If U has o.n. columns and is a 4×2 matrix, then $U^T U = I$.

(e) If U has o.n. columns and is a 4×2 matrix, then $UU^T = I$.

(f) If A is $n \times n$ and not invertible, then $\lambda = 0$ is an eigenvalue of A.

(g) Let

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$

Then the rank of AA^T is 2.

17. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be the normalized eigenvectors of $A^T A$, where A is $m \times n$.

(a) Show that if λ_i is a non-zero eigenvalue of $A^T A$, then it is also a non-zero eigenvalue of AA^T.

(b) True or false? The eigenvectors form an orthogonal basis of $\mathbb{R}^n$.

(c) Show that, if $\mathbf{x} \in \mathbb{R}^n$, then the i^{th} coordinate of $\mathbf{x}$ (with respect to the eigenvector basis) is $\mathbf{x}^T \mathbf{v}_i$.

(d) Let $\alpha_1, \ldots, \alpha_n$ be the coordinates of $\mathbf{x}$ with respect to $\mathbf{v}_1, \ldots, \mathbf{v}_n$.
Show that

$$\|\mathbf{x}\|_2 = \alpha_1^2 + \alpha_2^2 + \ldots + \alpha_n^2$$

I'll allow you to show it just using just two vectors, $\mathbf{v}_1, \mathbf{v}_2$.

(e) Show that $A\mathbf{v}_i \perp A\mathbf{v}_j$

(f) Show that $A\mathbf{v}_i$ is an eigenvector of AA^T.

18. Show that, for the line of best fit, the normal equations produce the same equations as minimizing an appropriate error function. To be more specific, set the data as $(x_1, t_1), \ldots, (x_p, t_p)$ and define the error function first. Minimize the error function to find the system of equations in m, b. Show this system is the same you get using the normal equations.

19. Given data:

x	-1	0	1
y	2	1	1

(a) Give the matrix equation for the *line of best fit.*

(b) Compute the normal equations.

(c) Solve the normal equations for the slope and intercept.

20. Use the data in Exercise (19) to find the parabola of best fit: $y = ax^2 + bx + c$. (NOTE: Will you only get a least squares solution, or an actual solution to the appropriate matrix equation?)

21. Let $\mathbf{x} = [1, 2, 1]^T$. Find the matrix $\mathbf{x}\mathbf{x}^T$, its eigenvalues, and eigenvectors. (Also, think about what happens in the general case, where a matrix is defined by $\mathbf{x}\mathbf{x}^T$). HINT: SVD

22. Suppose $\mathbf{x}$ is a vector containing n real numbers, and we understand that $m\mathbf{x} + b$ is Matlab-style notation (so we can add a vector to a scalar, done component-wise).

    (a) Find the mean of $\mathbf{y} = m\mathbf{x} + b$ in terms of the mean of $\mathbf{x}$.

    (b) Show that, for fixed constants a, b, $\mathrm{Cov}(\mathbf{x} + a, \mathbf{y} + b) = \mathrm{Cov}(\mathbf{x}, \mathbf{y})$

    (c) If $\mathbf{y} = m\mathbf{x} + b$, then find the covariance and correlation coefficient between $\mathbf{x}$ and $\mathbf{y}$.

23. Suppose we have a subspace W spanned by an orthonormal set of non-zero vectors, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, each is in $\mathbb{R}^{1000}$. If a vector $\mathbf{x}$ is in W, then there is a low dimensional (three dimensional in fact) representation of $\mathbf{x}$. What is it?

24. Consider the underdetermined "system of equations": $x+3y+4z = 1$. In matrix-vector form $A\mathbf{x} = \mathbf{b}$, write the matrix A first.

    (a) What is the dimension of each of the four fundamental subspaces?

    (b) Find bases for the four fundamental subspaces.

    (c) Find a solution to the equation with at least 2 zeros.

    (d) Find a 3×3 matrix P so that given a vector $\mathbf{x}$, $P\mathbf{x}$ is the projection of $\mathbf{x}$ into the row space of A.

25. (SVD) Given that the SVD of a matrix was given in Matlab as:

```
>> [U,S,V]=svd(A)
U =
   -0.4346   -0.3010    0.7745    0.3326   -0.1000
   -0.1933   -0.3934    0.1103   -0.8886   -0.0777
    0.5484    0.5071    0.6045   -0.2605   -0.0944
    0.6715   -0.6841    0.0061    0.1770   -0.2231
    0.1488   -0.1720    0.1502   -0.0217    0.9619
S =
    5.72          0            0
         0       2.89          0
         0          0          0
         0          0          0
         0          0          0
V =
    0.2321   -0.9483    0.2166
   -0.2770    0.1490    0.9493
    0.9324    0.2803    0.2281
```

(a) Which columns form a basis for the null space of A? For the column space of A? For the row space of A?

(b) How do we "normalize" the singular values? In this case, what are they (numerically)?

(c) What is the rank of A?

(d) How would you compute the pseudo-inverse of A (do not actually do it):

(e) Let B be formed using the first two columns of U. Would the matrix $B^T B$ have any special meaning? Would BB^T?

26. In computing the best basis, what did we mean by "best"? (Be as specific as you can).

27. Suppose we have p points in $\mathbb{R}^n$,

$$\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_p\}$$

and we project each point to some (fixed) unit vector $\mathbf{u}$. Show that the mean of the scalar projections is the same as the projection of the mean vector.

28. Continuing with the last problem, if we assume the mean of the data is zero, then show that the variance of the scalar projections to the unit vector $\mathbf{u}$ is given by:

$$\mathbf{u}^T \left(\frac{1}{p-1} \sum_{i=1}^{p} \mathbf{x}_i \mathbf{x}_i^T \right) \mathbf{u}$$

Hint: Recall that with our assumption of the mean, you can write the sample variance as $\displaystyle\sum_{i=1}^{p} (\mathbf{u}^T \mathbf{x}_i)^2$

29. Suppose $\lambda, \mathbf{v}$ are the eigenvalue and (unit) eigenvector of the symmetric matrix C. Simplify the expression: $\mathbf{v}^T C \mathbf{v}$

Continuing, if $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, where $\lambda_1, \mathbf{v}_1$ and $\lambda_2, \mathbf{v}_2$ are eigenvalue, unit eigenvector pairs for the symmetric matrix C, show that

$$\mathbf{x}^T C \mathbf{x} = \lambda_1 c_1^2 + \lambda_2 c_2^2$$