Week 10 Homework Summary

I. From Monday: 1, 4, 5 on p. 55:

1. Show that $\text{Null}(A^T) \perp \text{Col}(A)$.

4. Prove the Pythagorean Theorem by induction: Given a set of n orthogonal vectors $\{\boldsymbol{x}_i\}$

$$\|\sum_{i=1}^{n} \boldsymbol{x}_i\|_2^2 = \sum_{i=1}^{n} \|\boldsymbol{x}_i\|_2^2$$

The case where $n = 1$ is trivial, so you might look at $n = 2$ first. Try starting with

$$\|\mathbf{x} + \mathbf{y}\|^2 = (\mathbf{x} + \mathbf{y})^T(\mathbf{x} + \mathbf{y}) = \cdots$$

and then simplify to get $\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$. Now try the induction step on your own.

5. Let A be an $m \times n$ matrix where $m > n$, and let A have rank n. Let $\boldsymbol{y}, \hat{\boldsymbol{y}} \in \mathbb{R}^m$, such that $\hat{\boldsymbol{y}}$ is the orthogonal projection of $\boldsymbol{y}$ onto the column space of A. We want a formula for the matrix $\mathbb{P} : \mathbb{R}^m \to \mathbb{R}^m$ so that $\mathbb{P}\boldsymbol{y} = \hat{\boldsymbol{y}}$.

The following image shows the relevant subspaces (See page 56 for the graph)

(a) Why is the projector not $\mathbb{P} = AA^T$?

(b) Since $\hat{\boldsymbol{y}} - \boldsymbol{y}$ is orthogonal to the column space of A, show that

$$A^T(\hat{\boldsymbol{y}} - \boldsymbol{y}) = \mathbf{0} \tag{1}$$

(c) Show that there exists $\boldsymbol{x} \in \mathbb{R}^n$ so that Equation (1) can be written as:

$$A^T(A\boldsymbol{x} - \boldsymbol{y}) = 0 \tag{2}$$

(d) Argue that A^TA (which is $n \times n$) is invertible, so that Equation (2) implies that

$$\boldsymbol{x} = \left(A^TA\right)^{-1} A^T\boldsymbol{y}$$

(e) Finally, show that this implies that

$$\mathbb{P} = A\left(A^TA\right)^{-1} A^T$$

Note: If A has rank $k \neq n$, then we will need something different, since A^TA will not be full rank. The missing piece is the singular value decomposition, to be discussed later.

II. From Tuesday: 1, 3, 4, 6, 7, 8 on p. 53 (Yes, we were working a little backwards).

1. Let the subspace H be formed by the span of the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ given below. Given the point $\boldsymbol{x}_1, \boldsymbol{x}_2$ below, find which one belongs to H, and if it does, give its coordinates. (NOTE: The basis vectors are NOT orthonormal)

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad \boldsymbol{x}_1 = \begin{bmatrix} 7 \\ 4 \\ 0 \end{bmatrix} \quad \boldsymbol{x}_2 = \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}$$

3. Let the subspace G be the plane defined below, and consider the vector $\boldsymbol{x}$, where:

$$G = \left\{ \alpha_1 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \text{ such that } \alpha_1, \alpha_2 \in \mathbb{R} \right\} \qquad \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

(Skip part (a))

b. Find the orthogonal projection of the given $\boldsymbol{x}$ onto the plane G (note that the vectors are orthogonal).
(Added) Show that the formula in 5(e) from the previous page works for this problem as well.

c. Find the distance from the plane G to the vector $\boldsymbol{x}$.

4. If the low dimensional representation of a vector $\boldsymbol{x}$ is $[9, -1]^T$ and the basis vectors are $[1, 0, 1]^T$ and $[3, 1, 1]^T$, then what was the original vector $\boldsymbol{x}$? (HINT: it is easy to compute it directly)

6. Let $\boldsymbol{a} = [-1, 3]^T$. Find a square matrix P so that $P\boldsymbol{x}$ is the orthogonal projection of $\boldsymbol{x}$ onto the span of $\boldsymbol{a}$.

7. (Reworded from the text) Suppose we have vectors $\mathbf{x}$, $\mathbf{u}$, and the projection of $\mathbf{x}$ onto $\mathbf{u}$ (Call it $\hat{\mathbf{x}}$).
   - If the projection is *orthogonal*, then which vectors should be perpendicular?
   - If $\hat{\mathbf{x}} = c\mathbf{u}$, find the constant c from your previous answer.

8. Continuing with the last exercise, show that $UU^T\mathbf{x}$ is the *orthogonal* projection of $\mathbf{x}$ into the space spanned by the columns of U by showing that $(UU^T\boldsymbol{x} - \boldsymbol{x})$ is orthogonal to $\mathbf{u}_i$ for any $i = 1, 2, \cdots, k$.

III. Homework from Thursday, Nov 6th

1 . Assume that A and B are row equivalent, where:

$$A = \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ -2 & -3 & 1 & -1 & -5 \\ -3 & -4 & 0 & -2 & -3 \\ 3 & 6 & -6 & 5 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 4 & 0 & -3 \\ 0 & 1 & -3 & 0 & 5 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) State which vector space ($\mathbb{R}^?$) contains each of the four subspaces, and state the dimension of each of the four subspaces:

(b) Find a basis for Col(A):

(c) Find a basis for Row(A):

(d) Find a basis for Null(A):

2. From Monday's HW, we said that if we're solving for $A\mathbf{x} = \mathbf{y}$ and A has linearly independent columns, then we can solve for $\mathbf{x}$ using the **normal equations** (just as we did in the linear regression problem):

$$A\mathbf{x} = \mathbf{y} \quad \Rightarrow \quad A^T A\mathbf{x} = A^T\mathbf{y}$$

And since A has full rank (it has linearly independent columns), then

$$\mathbf{x} = (A^T A)^{-1} A^T \mathbf{y}$$

The matrix $(A^T A)^{-1} A^T$ is called the *pseudoinverse* of A, and is denoted by $A^\dagger$. Find the pseudoinverse matrix of A, if

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

3. Also from Monday's HW, we said that if

$$\mathbf{x} = (A^T A)^{-1} A^T \mathbf{y}$$

then

$$A\mathbf{x} = A(A^T A)^{-1} A^T \mathbf{y}$$

Since $A\mathbf{x}$ is in the column space of A, then the vector on the right side of the equation must be the orthogonal projection of $\mathbf{y}$ into the column space of A. Therefore, the matrix that takes $\mathbf{y}$ and produces the projection of $\mathbf{y}$ is:

$$P = A(A^T A)^{-1} A^T$$

Find the matrix P using the matrix A from the previous exercise.