Week 10 Homework Summary

I. From Monday: 1, 4, 5 on p. 55:

1. Show that Null$(A^T) \perp$ Col(A).
   SOLUTION: Let $B = A^T$. Then we prove that Null$(B) \perp$ Row(B) which is equivalent.
   If we write B in terms of rows, $B = [r_1, r_2, \dots, r_m]^T$, then the i^{th} component of $B\boldsymbol{x}$ can be written as $\boldsymbol{r}_i\boldsymbol{x} = 0$. Thus, if $\boldsymbol{x}$ is in the nullspace of B, then it is orthogonal to every row of B.

4. Prove the Pythagorean Theorem by induction: Given a set of n orthogonal vectors $\{\boldsymbol{x}_i\}$

$$\|\sum_{i=1}^{n} \boldsymbol{x}_i\|_2^2 = \sum_{i=1}^{n} \|\boldsymbol{x}_i\|_2^2$$

   The case where $n = 1$ is trivial, so you might look at $n = 2$ first. Try starting with

$$\|\mathbf{x} + \mathbf{y}\|^2 = (\mathbf{x} + \mathbf{y})^T(\mathbf{x} + \mathbf{y}) = \cdots$$

   and then simplify to get $\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$. Now try the induction step on your own.
   SOLUTION:
   The case with $n = 1$ is trivial. The case with $n = 2$ is straightforward. Assume true for $k = n$,

$$\|\sum_{i=1}^{n} \boldsymbol{x}_i\|^2 = \sum_{i=1}^{n} \|\boldsymbol{x}_i\|^2$$

   Show true for $k = n + 1$:

$$\|\sum_{i=1}^{n+1} \boldsymbol{x}_i\|^2 = \left(\sum_{i=1}^{n} \boldsymbol{x}_i + \boldsymbol{x}_{n+1}\right)^T \left(\sum_{i=1}^{n} \boldsymbol{x}_i + \boldsymbol{x}_{n+1}\right) =$$

$$\left(\sum_{i=1}^{n} \boldsymbol{x}_i\right)^T \left(\sum_{i=1}^{n} \boldsymbol{x}_i\right) + \left(\sum_{i=1}^{n} \boldsymbol{x}_i\right) \boldsymbol{x}_{n+1}^T + \boldsymbol{x}_{n+1} \left(\sum_{i=1}^{n} \boldsymbol{x}_i\right)^T + \boldsymbol{x}_{n+1}^T \boldsymbol{x}_{n+1}$$

   By the induction step, that first term is:

$$\left(\sum_{i=1}^{n} \boldsymbol{x}_i\right)^T \left(\sum_{i=1}^{n} \boldsymbol{x}_i\right) = \sum_{i=1}^{n} \|\mathbf{x}_i\|^2$$

   The second and third terms are zero since $\mathbf{x}_{n+1}$ is orthogonal to the first n vectors, and the last term is $\|\mathbf{x}_{n+1}\|^2$. Therefore, the expression simplifies to the desired value,

$$\|\sum_{i=1}^{n+1} \boldsymbol{x}_i\|^2 = \sum_{i=1}^{n+1} \|\boldsymbol{x}_i\|^2$$

5. Let A be an $m \times n$ matrix where $m > n$, and let A have rank n. Let $\boldsymbol{y}, \hat{\boldsymbol{y}} \in \mathbb{R}^m$, such that $\hat{\boldsymbol{y}}$ is the orthogonal projection of $\boldsymbol{y}$ onto the column space of A. We want a formula for the matrix $\mathbb{P} : \mathbb{R}^m \to \mathbb{R}^m$ so that $\mathbb{P}\boldsymbol{y} = \hat{\boldsymbol{y}}$.

The following image shows the relevant subspaces (See page 56 for the graph)

(a) Why is the projector not $\mathbb{P} = AA^T$?
SOLUTION: If the columns of A were orthonormal, it would be. In class, we found the actual projection matrix to be the one below, which is:

$$P = A(A^TA)^{-1}A^T$$

which we are showing now.

(b) Since $\hat{\boldsymbol{y}} - \boldsymbol{y}$ is orthogonal to the column space of A, show that

$$A^T(\hat{\boldsymbol{y}} - \boldsymbol{y}) = \boldsymbol{0} \tag{1}$$

SOLUTION: We know that $\text{Col}(A) \perp \text{Null}(A^T)$. Therefore, if $\hat{\boldsymbol{y}} - \boldsymbol{y}$ is orthogonal to the column space of A, then it must be in the null space of A^T.

(c) Show that there exists $\boldsymbol{x} \in \mathbb{R}^n$ so that Equation (1) can be written as:

$$A^T(A\boldsymbol{x} - \boldsymbol{y}) = 0 \tag{2}$$

SOLUTION: Since $\hat{\boldsymbol{y}}$ is in the columnspace of A, by definition there exists a $\boldsymbol{x} \in \mathbb{R}^n$ such that $A\boldsymbol{x} = \hat{\boldsymbol{y}}$.

(d) Argue that A^TA (which is $n \times n$) is invertible, so that Equation (2) implies that

$$\boldsymbol{x} = \left(A^TA\right)^{-1} A^T\boldsymbol{y}$$

SOLUTION: The matrix A has rank n, therefore A^TA is an $n \times n$ matrix with rank n as well (the ranks are equal). Therefore, A^TA is invertible, and we get the **pseudoinverse**, $A^\dagger = (A^TA)^{-1}A^T$.

(e) Finally, show that this implies that

$$\mathbb{P} = A\left(A^TA\right)^{-1} A^T$$

SOLUTION: Multiply both sides of the previous equation by A.

II. From Tuesday: 1, 3, 4, 6, 7, 8 on p. 53 (Yes, we were working a little backwards).

1. Let the subspace H be formed by the span of the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ given below. Given the point $\boldsymbol{x}_1, \boldsymbol{x}_2$ below, find which one belongs to H, and if it does, give its coordinates. (NOTE: The basis vectors are NOT orthonormal)

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad \boldsymbol{x}_1 = \begin{bmatrix} 7 \\ 4 \\ 0 \end{bmatrix} \quad \boldsymbol{x}_2 = \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}$$

SOLUTION: Rather than row-reduce twice, we'll do it once on the augmented matrix below.

$$\left[\begin{array}{rr|rr} 1 & 2 & 7 & 4 \\ 2 & -1 & 4 & 3 \\ -1 & 1 & 0 & -1 \end{array}\right] \to \left[\begin{array}{rr|rr} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

How should this be interpreted? The second vector, $\boldsymbol{x}_2$ is in H, as it can be expressed as $2\boldsymbol{v}_1 + \boldsymbol{v}_2$. Its low dimensional representation is thus $[2, 1]^T$.

The first vector, $\boldsymbol{x}_1$, cannot be expressed as a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, so it does not belong to H.

3. Let the subspace G be the plane defined below, and consider the vector $\boldsymbol{x}$, where:

$$G = \left\{ \alpha_1 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \text{ such that } \alpha_1, \alpha_2 \in \mathbb{R} \right\} \qquad \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

(Skip part (a))

b. Find the orthogonal projection of the given $\boldsymbol{x}$ onto the plane G (note that the vectors are orthogonal).

SOLUTION: Let the vectors given (normalize them) be $\mathbf{u}_1$ and $\mathbf{u}_2$. Then we compute:

$$\mathrm{Proj}_G(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{x} \cdot \mathbf{u}_2)\mathbf{u}_2 = \frac{-3}{14} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + \frac{3}{10} \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \approx \begin{bmatrix} 0.686 \\ -0.943 \\ 0.429 \end{bmatrix}$$

(Added) Show that the formula in 5(e) from the previous page works for this problem as well.

SOLUTION: I shouldn't have added this- This was enough computation.

c. Find the distance from the plane G to the vector $\boldsymbol{x}$.

SOLUTION: The distance is the length of the vector going from $\mathbf{x}$ to the projection of $\mathbf{x}$ in G, or

$$\left\| \begin{array}{r} 1 - 0.686 \\ 0.943 \\ 2 - 0.429 \end{array} \right\| \approx 1.859$$

4. If the low dimensional representation of a vector $\boldsymbol{x}$ is $[9,-1]^T$ and the basis vectors are $[1,0,1]^T$ and $[3,1,1]^T$, then what was the original vector $\boldsymbol{x}$? (HINT: it is easy to compute it directly)
SOLUTION: $[6,-1,8]^T$.

6. Let $\boldsymbol{a} = [-1,3]^T$. Find a square matrix P so that $P\boldsymbol{x}$ is the orthogonal projection of $\boldsymbol{x}$ onto the span of $\boldsymbol{a}$.
SOLUTION: We're thinking of the outer product divided by the inner product, or

$$P = \frac{1}{10}\begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$$

7. (Reworded from the text) Suppose we have vectors $\mathbf{x}$, $\mathbf{u}$, and the projection of $\mathbf{x}$ onto $\mathbf{u}$ (Call it $\hat{\mathbf{x}}$).
   - If the projection is *orthogonal*, then which vectors should be perpendicular?
     SOLUTION: The vector $\mathbf{x}-\hat{\mathbf{x}}$ should be orthogonal to $\mathbf{u}$.
   - If $\hat{\mathbf{x}} = c\mathbf{u}$, find the constant c from your previous answer.
     SOLUTION: The constant comes from the orthogonality

$$(\mathbf{x} - c\mathbf{u})\cdot\mathbf{u} = 0 \quad \Rightarrow \quad \mathbf{x}\cdot\mathbf{u} = c\mathbf{u}\cdot\mathbf{u} \quad \Rightarrow \quad c = \frac{\mathbf{x}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}$$

8. Continuing with the last exercise, show that $UU^T\mathbf{x}$ is the *orthogonal* projection of $\mathbf{x}$ into the space spanned by the columns of U by showing that $(UU^T\boldsymbol{x} - \boldsymbol{x})$ is orthogonal to $\mathbf{u}_i$ for any $i = 1,2,\cdots,k$.
SOLUTION: Notice that

$$UU^T = \mathbf{u}_1\mathbf{u}_1^T + \mathbf{u}_2\mathbf{u}_2^T + \cdots + \mathbf{u}_k\mathbf{u}_k^T$$

so that

$$(UU^T\mathbf{x}-\mathbf{x})\cdot\mathbf{u}_i = [(0+0+\cdots+0) + (\mathbf{u}_i\cdot\mathbf{u}_i)(\mathbf{u}_i\cdot\mathbf{x}) + (0+0+\cdots+0)] - \mathbf{x}\cdot\mathbf{u}_i = 0$$

III. Homework from Thursday, Nov 6th

1 . Assume that A and B are row equivalent, where:

$$A = \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ -2 & -3 & 1 & -1 & -5 \\ -3 & -4 & 0 & -2 & -3 \\ 3 & 6 & -6 & 5 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 4 & 0 & -3 \\ 0 & 1 & -3 & 0 & 5 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) State which vector space ($\mathbb{R}^?$) contains each of the four subspaces, and state the dimension of each of the four subspaces:
SOLUTION: Here are the spaces and dimensions:

A. The row space of A is a subspace of $\mathbb{R}^5$ and has dimension 3 (this is the number of pivot rows).

B. The column space of A is a subspace of $\mathbb{R}^4$ and has dimension 3 (this is also known as the rank of A and is the number of pivot columns).

C. The null space of A is a subspace of $\mathbb{R}^5$ and has dimension 2 (the number of free variables).

D. The null space of A^T is a subspace of $\mathbb{R}^4$ and has dimension 1 (because the dimension of the column space is 3, and they should add to 4).

(b) Find a basis for Col(A):
SOLUTION: We see from the RREF that the pivot columns are cols 1, 2, and 4. Be sure to use the columns from the original matrix A! The basis is the set

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -4 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \\ 5 \end{bmatrix} \right\}$$

(c) Find a basis for Row(A):
SOLUTION: We can use the reduced rows (written as columns) for the basis:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 4 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -4 \end{bmatrix} \right\}$$

(d) Find a basis for Null(A):
SOLUTION: Solve $A\mathbf{x} = \mathbf{0}$, and the vectors are the basis vectors:

$$\begin{aligned} x_1 &= -4x_3 + 3x_5 \\ x_2 &= 3x_3 - 5x_5 \\ x_3 &= x_3 \\ x_4 &= \qquad 4x_5 \\ x_5 &= \qquad\ \ x_5 \end{aligned} \quad \Rightarrow \left\{ \begin{bmatrix} -4 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}$$

2. From Monday's HW, we said that if we're solving for $A\mathbf{x} = \mathbf{y}$ and A has linearly independent columns, then we can solve for $\mathbf{x}$ using the **normal equations** (just as we did in the linear regression problem):

$$A\mathbf{x} = \mathbf{y} \quad \Rightarrow \quad A^T A\mathbf{x} = A^T\mathbf{y}$$

And since A has full rank (it has linearly independent columns), then

$$\mathbf{x} = (A^T A)^{-1} A^T \mathbf{y}$$

The matrix $(A^T A)^{-1} A^T$ is called the *pseudoinverse* of A, and is denoted by $A^\dagger$. Find the pseudoinverse matrix of A, if

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

SOLUTION: This is just a matter of multiplying it out. See the exam review sheet.

3. Also from Monday's HW, we said that if

$$\mathbf{x} = (A^T A)^{-1} A^T \mathbf{y}$$

then

$$A\mathbf{x} = A(A^T A)^{-1} A^T \mathbf{y}$$

Since $A\mathbf{x}$ is in the column space of A, then the vector on the right side of the equation must be the orthogonal projection of $\mathbf{y}$ into the column space of A. Therefore, the matrix that takes $\mathbf{y}$ and produces the projection of $\mathbf{y}$ is:

$$P = A(A^T A)^{-1} A^T$$

Find the matrix P using the matrix A from the previous exercise.

SOLUTION: It's just a matter of multiplying it out. See the exam review sheet.