Exercise Set 3

This includes the relevent exercises handed out earlier with some deletions and some additions. For the direction field/solution curve plots, use the website linked from our class site.

1. Verify that the following function solves the given system of DEs:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \mathrm{e}^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}$$

2. For each matrix, find the eigenvalues and eigenvectors. Plot the direction field with some solution curves (for the two dimensional systems), then give the solution to $\mathbf{x}' = A\mathbf{x}$ in each case.

   (a) $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$

   (c) $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

   (d) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

   (e) $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}$

3. For each given λ and $\mathbf{v}$, find an expression for $\mathrm{Re}(\mathrm{e}^{\lambda t}\mathbf{v})$ and $\mathrm{Im}(\mathrm{e}^{\lambda t}\mathbf{v})$:

   (a) $\lambda = 3i$, $\mathbf{v} = [1 - i, 2i]^T$

   (b) $\lambda = 1 + i$, $\mathbf{v} = [i, 2]^T$

   (c) $\lambda = 2 - i$, $\mathbf{v} = [1, 1 + 2i]^T$

   (d) $\lambda = i$, $\mathbf{v} = [2 + 3i, 1 + i]^T$

4. (Extra Practice) For each system below, find y as a function of x by first writing the differential equation as dy/dx.

   (a) $\begin{aligned} x' &= -2x \\ y' &= y \end{aligned}$

   (b) $\begin{aligned} x' &= y + x^3 y \\ y' &= x^2 \end{aligned}$

   (c) $\begin{aligned} x' &= -(2x + 3) \\ y' &= 2y - 2 \end{aligned}$

   (d) $\begin{aligned} x' &= -2y \\ y' &= 2x \end{aligned}$

5. Use the Poincaré diagram to determine how changing a in the system below will change the equilibrium. Each system is $\mathbf{x}' = A\mathbf{x}$, where A is given below.

   (a) $A = \begin{bmatrix} a & -1 \\ 2 & 0 \end{bmatrix}$

   (b) $A = \begin{bmatrix} a & 1 \\ a & a \end{bmatrix}$

6. Lab: Bifurcations in Linear Systems.

   Discussion: A "bifurcation" is the change in the number or type of equilibrium solutions we have. For example, if the matrix A is invertible, we know that $\mathbf{x}' = A\mathbf{x}$ has only one equilibrium- At the origin. If there is a change so that the matrix becomes non-invertible (or singular), then there are an infinite number of equilbria.

   Suppose we consider the linear system:

$$\begin{aligned} x' &= ax + by \\ y' &= -x - y \end{aligned}$$

   where a, b can take on any real number. Write a report addressing the following:

   For each value of a, b, classify the linear system as a source, sink, center, spiral sink (or source), or degenerate sink (or source). Draw a picture of the (a, b) plane and indicate the values of a, b for which the system is of each type. Be sure to describe all of the computations involved in creating this picture.

The Poincaré Diagram

The Poincaré Diagram is a nice way of classifying the equilibrium for all linear two dimensional systems. Here is a quick introduction to it:

Given $\mathbf{x}' = A\mathbf{x}$, the characteristic equation is given by

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0 \quad \Rightarrow \quad \lambda = \frac{\mathrm{Tr}(A) \pm \sqrt{\Delta}}{2} = \frac{\mathrm{Tr}(A) \pm \sqrt{\mathrm{Tr}(A)^2 - 4\det(A)}}{2}$$

The type of eigenvalues we get depends on the discriminant. The key value is where the discriminant is zero:

$$\Delta = \mathrm{Tr}(A)^2 - 4\det(A) = 0$$

Now, thinking of the trace being on the "x-axis" and the determinant being on the "y-axis", we see the parabola:

$$x^2 - 4y = 0$$

This divides the trace-determinant plane into different regions, and we can classify the equilibrium in each region. Here is an example exercise using this:

Example:

Let $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} a & a^2 + a \\ 1 & a \end{bmatrix}$. Describle how changing the value of a changes the type/number of equilibrium solutions.

SOLUTION: For the Poincaré diagram, we need the trace, determinant and discriminant:

$$\mathrm{Tr}(A) = 2a \qquad \det A = a^2 - (a^2 + a) = -a \qquad \Delta = (2a)^2 - 4(-a) = 4a^2 + 4a = 4a(a+1)$$

To see where we are in the diagram, we need to know the signs of these three values, so we can perform a sign chart analysis. We break the number line up where each quantity is zero- In this case, $a = 0$ or $a = -1$:

$\mathrm{Tr}(A) = 2a$	$-$	$-$	$-$	0	$+$
$\det(A) = -a$	$+$	$+$	$+$	0	$-$
$\Delta = 4a(a+1)$	$+$	0	$-$	0	$+$
	$a < -1$	$a = -1$	$-1 < a < 0$	$a = 0$	$a > 0$
	Sink	Degen Sink	Spiral Sink	Unif Motion	Saddle