Difference Equations
M350 Class Notes

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1 Introduction

- The simplest difference equation:

$$x_{n+1} = ax_n \qquad \text{or} \qquad x_{n+1} - ax_n = 0$$

- We might add some complexity by adding a constant or a term that depends on n:

$$x_{n+1} = ax_n + b \qquad x_{n+1} = ax_n + b_n$$

- We say that $\{x_0, x_1, x_2, \cdots\}$ is a SOLUTION to the difference equation if the sequence satisfies the given difference equation. Typically, we do not write solutions this way- We typically define a solution in terms of some formula depending on n:

$$x_n = f(n)$$

- A second order, linear, homogeneous difference equation is of the form:

$$x_{n+2} + ax_{n+1} + bx_n = 0$$

It is second order because of the x_{n+2}, and it is linear since x_{n+2}, x_{n+1}, x_n all appear as linear terms. The reason it is homogeneous is because we do not have b or b_n on the right hand side.

- The value x^* is said to be a fixed point (or equilibrium) to a difference equation given by: $x_{n+1} = f(x_n)$ if $x^* = f(x^*)$.

2 First Order Homogeneous DE

Given a first order homogeneous difference equation,

$$x_{n+1} = ax_n$$

we can say that $f(n) = a^n x_0$ is the solution. The fixed point to this equation is where:

$$x = ax$$

so $x = 0$ is the only fixed point (unless $a = 1$, in which case every point is fixed). If $|a| > 1$, all solutions will tend (in size) to infinity. If $|a| < 1$, all solutions will tend to zero as n becomes large.

3 First Order, Nonhomogeneous DE, Part I

Given a first order nonhomogeneous difference equation of the form:

$$x_{n+1} = ax_n + b$$

we can transform it into a first order homogeneous DE. To do this, note that the only fixed point to a first order homogeneous DE is zero. But our difference equation has a different fixed point:

$$x = ax + b \quad \Rightarrow \quad (1-a)x = b \quad \Rightarrow \quad x = \frac{b}{1-a}, a \neq 1$$

If $a = 1$, then the difference equation has the form:

$$x_{n+1} = x_n + b \quad \Rightarrow \quad x_n = x_0 + nb, b \neq 0$$

In this case, $|x_n| \to \infty$ as n gets large, so we'll focus on the other case, where $a \neq 1$. In this case, we found the fixed point, $x^* = \frac{b}{1-a}$. We create a new sequence, y_n, so that:

$$y_n = x_n - x^*$$

Now the difference equation associated to y_n will have a fixed point at $y^* = 0$, and:

$$y_{n+1} = x_{n+1} - x^* = ax_n + b - x^* = a(y_n + x^*) + b - x^* = ay_n - (1-a)x^* + b = ay_n$$

So the closed form solution is: $y_n = a^n y_0$, or:

$$x_n - x^* = a^n(x_0 - x^*) \quad \Rightarrow \quad x_n = a^n(x_0 - x^*) + x^*$$

Which gives the closed form solution to the first order, nonhomogeneous difference equation. It could be written in a different way:

$$x_n = a^n x_0 + b\,\frac{1-a^n}{1-a}$$

The last term is the sum of a partial geometric series in a.

4 Summary so far

In the first two types of difference equations, we can classify the long term behavior of x_n rather simply: The long term behavior will converge to a fixed point if $|a| < 1$. The long term behavior will diverge if $|a| > 1$. If $a = 1$, the long term behavior could be fixed for all x_0 (if $b = 0$) or diverge (if $b \neq 0$). If $a = -1$, the behavior will oscillate between $\pm x_0$ (if $b = 0$), or oscillate between x_0 and $-x_0 + b$ (if $b \neq 0$)- the second case you'll consider in the exercises.

5 First Order, Nonhomogeneous DE, Part II

In this case, we consider difference equations of the form:

$$x_{n+1} = ax_n + b_n$$

where b_n now depends on n. For example, we might have something like:

$$x_{n+1} = 3x_n - 2n + 1$$

Before going into the details of the solution, let's consider a general situation. Suppose I've got a difference equation of the form we are considering, and suppose I know a particular solution, p_n. We show that, in this case, anything of the form:

$$q_n = ca^n + p_n$$

is *also* a solution:

We need to show that q_n is a solution to the difference equation. This means that we need to show that:

$$q_{n+1} = aq_n + b_n$$

On the left hand side of the equation, we have:

$$q_{n+1} = ca^{n+1} + p_{n+1}$$

and since p_n is a solution to the difference equation,

$$q_{n+1} = ca^{n+1} + ap_n + b_n = a(ca^n + p_n) + b_n$$

Now on the right hand side of the equation,

$$aq_n + b_n = a(ca^n + p_n) + b_n$$

Therefore, we have shown that $ca^n + p_n$ is also a solution. Now we have two things to do to solve a first order nonhomogeneous difference equation: Find p_n, and determine the value of c- the value of c will depend on the initial condition, x_0. Before we do this, consider the following difference equations together with their particular solutions:

$$\begin{array}{ll} x_{n+1} = 3x_n + (3n-1)2^n & p_n = (-3n-5)2^n \\ x_{n+1} = 3x_n + (3n-1)3^n & p_n = n\left(\frac{1}{2}n - \frac{5}{6}\right)3^n \\ x_{n+1} = 3x_n + (1+n+2n^2) & p_n = -(\frac{7}{4} + \frac{3}{2}n + n^2) \end{array}$$

You might be seeing a pattern- If b_n is of a particular form, then we can make an ansatz[1] about the form of the particular solution. For example, if b_n has the form:

$$b_n = c_0 + c_1 n + \ldots + c_m n^m \Rightarrow p_n = A_0 + A_1 n + \ldots + A_m n^m$$

or,

$$b_n = (c_0 + c_1 n + \ldots + c_m n^m)\, r^n \Rightarrow p_n = (A_0 + A_1 n + \ldots + A_m n^m) r^n, r \neq a$$

at which point we would need to solve for the $A_0, A_1, \ldots, A_m$. Let's do this with the first example, since we know what the answer should be. Given $x_{n+1} = 3x_n + (3n-1)2^n$, my ansatz will be: $p_n = (A + Bn)2^n$. Inserting this into the left hand side of the equation,

$$p_{n+1} = (A + B(n+1))2^{n+1}$$

and into the right hand side of the equation,

$$3p_n + (3n-1)2^n = 3(A + Bn)2^n + (3n-1)2^n$$

Now equate both sides:

$$(A + B(n+1))2^{n+1} = 3(A + Bn)2^n + (3n-1)2^n \quad \Rightarrow \quad 2(A + Bn + B) = 3A + 3Bn + 3n - 1$$

To solve this, we make an observation: If two polynomials are equal *for all* input values, then the coefficients must be equal. In this case, we will equate the coefficients of n and the (separately) the constants. This will give two equations in our two unknowns:

$$\begin{array}{lrl} \text{Coeffs of } n & 2B & = 3B + 3 \\ \text{Constants} & 2A + 2B & = 3A - 1 \end{array}$$

From which we get: $p_n = -(5 + 3n)2^n$. We can check our answer by checking that it does indeed satisfy the difference equation:

$$p_{n+1} = -(5 + 3(n+1))2^{n+1} = (-5 - 3n - 3)2^{n+1} = (-8 - 3n)2^{n+1}$$

and

$$3p_n + (3n-1)2^n = 3(-(5 + 3n)2^n) + (3n-1)2^n = (-8 - 3n)2^{n+1}$$

Similarly, we can solve more complex difference equations, but the algebra gets a little messy. In these cases, Maple can also give solutions using `rsolve` (for recurrence solver). Here's an example in Maple:

[1] Ansatz is a German word that has many translations- "basic approach" or "point of departure", for example. In mathematics, we see it used in differential equations to mean a model form for a solution; an "educated guess", if you will.

```
> eqn:=x(n+1)=3*x(n)+(3*n-1)*2^n;
                                                  n
            eqn := x(n + 1) = 3 x(n) + (3 n - 1) 2

> rsolve({eqn},x(n));
                     n      n             n      n
             x(0) 3  + 5 3  + (-3 n - 3) 2  - 2 2
```

To solve a difference equation with an initial condition, there's a slight change. To solve $x_{n+1} = -x_n + \mathrm{e}^{-2n}$ with an initial condition $x(0) = 1$:

```
> eqn:=x(n+1)=-x(n)+exp(-2*n);
                eqn := x(n + 1) = -x(n) + exp(-2 n)
> rsolve({eqn,x(0)=1},x(n));
               n                      n
           (-1)  - 1/2 exp(-2 n) (-1)  + 1/2 exp(-2 n)
```

5.1 Summary of First Order, Nonhomogeneous DE

Given: $x_{n+1} = ax_n + b_n$, and a particular solution p_n, then the closed form general solution is:

$$x_n = ca^n + p_n$$

where c can be solved, given an initial condition. The particular solution, p_n, can be solved using an ansatz as long as b_n is of a particular form. We can also use Maple to solve the difference equation. In these cases, the long term behavior might be simple (convergence to a point), or might be more complicated. If you've had differential equations, you might compare this section with first order nonhomogeneous differential equations of the form:

$$y' + ay = f(t)$$

where we solve for the nonhomogeneous part of the solution by again using an ansatz based on the form of f.

6 Second Order, Homogeneous DE

Here we consider second order, linear, homogeneous difference equations. These are of the form:

$$x_{n+2} + \alpha x_{n+1} + \beta x_n = 0$$

Use an ansatz of $x_n = \lambda^n$ to back-substitute and solve for λ.

$$\lambda^{n+2} + \alpha\lambda^{n+1} + \beta\lambda^n = 0$$

$$\lambda^n \left(\lambda^2 + \alpha\lambda + \beta\right) = 0$$

Therefore, either $\lambda = 0$ or from the quadratic formula,

$$\lambda = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}$$

From this, there are three cases, depending on the discriminant $\alpha^2 - 4\beta$, which is positive, zero, or negative.

- If the discriminant is positive, we have two distinct real values of λ. The general solution is given by:

  $$x_n = C_1\lambda_1^n + C_2\lambda_2^n$$

  The values of C_1, C_2 are generally found by defining x_0 and x_1. We could write these constants out, and solve:

  $$x_0 = C_1 + C_2 \qquad x_1 = C_1\lambda_1 + C_2\lambda_2$$

  so we get two equations in the two unknowns.

- If the discriminant is zero, we have only 1 real λ, and the general solution is:

  $$x_n = C_1\lambda^n + nC_2\lambda^n = \lambda^n(C_1 + nC_2)$$

  Again, the values of C_1, C_2 can be found by defining x_0, x_1:

  $$x_0 = C_1 \qquad x_1 = \lambda(C_1 + C_2)$$

- Finally, if the discriminant is negative, we have two complex conjugate solutions,

  $$\lambda = -\frac{\alpha}{2} \pm \frac{\sqrt{4\beta - \alpha^2}}{2}\, i = a \pm bi$$

  We will define r as the size (or modulus) of the complex number,

  $$r = \sqrt{a^2 + b^2}$$

  and θ as the argument of the complex number,

  $$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

  You can visualize a complex number in the plane, where the real part is along the $x-$ axis, and the imaginary part is along the $y-$axis. The point $a + bi$ is associated with the ordered pair (a, b), and from this we have a triangle with side lengths a, b and hypotenuse r. The value of θ is always measured from the positive real axis. To remove any ambiguities introduced by the inverse tangent, we could take this to be the four-quadrant inverse tangent. For example, if $(a, b) = (1, 1)$ then $\theta = \frac{\pi}{4}$. If $(a, b) = (-1, 1)$, *theta* $= \frac{3\pi}{4}$. If $(a,b) = (-1,-1)$, then $\theta = \frac{5\pi}{4}$, and lastly, if $(a,b) = (1,-1)$, then $\theta = \frac{7\pi}{4}$. Furthermore, it won't matter if you consider $a + bi$ or $a - bi$, as the values of the arbitrary constants (see below) will change to make the solutions match up. In these cases, the general solution can be written as:

$$x_n = r^n \left(C_1 \cos(n\theta) + C_2 \sin(n\theta)\right)$$

Example: $x_{n+2} - 2x_{n+1} + 2x_n = 0$

In this case, the characteristic equation is:

$$\lambda^2 - 2\lambda + 2 = 0 \quad \lambda = 1 \pm i$$

Considering $\lambda = 1 + i$, $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$. This gives:

$$x_n = 2^{\frac{n}{2}} \left(C_1 \cos\left(\frac{n\pi}{4}\right) + C_2 \sin\left(\frac{n\pi}{4}\right)\right)$$

On the other hand, if $\lambda = 1 - i$, then $r = \sqrt{2}$ and $\theta = -\frac{\pi}{4}$, and

$$\hat{x}_n = 2^{\frac{n}{2}} \left(C_1 \cos\left(-\frac{n\pi}{4}\right) + C_2 \sin\left(-\frac{n\pi}{4}\right)\right) = 2^{\frac{n}{2}} \left(C_1 \cos\left(\frac{n\pi}{4}\right) - C_2 \sin\left(\frac{n\pi}{4}\right)\right)$$

Solving for C_1, C_2 in the first case, we get:

$$x_n = 2^{\frac{n}{2}} \left(x_0 \cos\left(\frac{n\pi}{4}\right) + (x_1 - x_0) \sin\left(\frac{n\pi}{4}\right)\right)$$

And in the second case,

$$\hat{x}_n = 2^{\frac{n}{2}} \left(x_0 \cos\left(\frac{n\pi}{4}\right) - (x_0 - x_1) \sin\left(\frac{n\pi}{4}\right)\right)$$

so that it did not matter if we took $\lambda = a + bi$ or $\lambda = a - bi$ as the "primary" complex number for our calculations.

In this last case, if $r < 1$, the solution will "spiral" towards the origin. If $r > 1$, the solution will "spiral" out.

In all cases, if we take a simple nonhomogeneous case,

$$x_{n+2} + \alpha x_{n+1} + \beta x_n = \gamma$$

then we can do a similar transformation as before. Find the fixed point,

$$x + \alpha x + \beta x = \gamma \qquad x^* = \frac{\gamma}{1 + \alpha + \beta}$$

Now, let $y_n = x_n - x^*$. We can show that this transformation leads to a homogeneous equation in y_n:

$$y_{n+2} + \alpha y_{n+1} + \beta y_n = 0$$

Systems of Difference Equations

We create systems of difference equations just as we did systems of differential equations. In the case of two variables, we might have:

$$\begin{aligned} a_{n+1} &= f(a_n, b_n) \\ b_{n+1} &= g(a_n, b_n) \end{aligned}$$

Where we again define the equilibrium solution as a point (a, b) such that

$$\begin{aligned} a &= f(a, b) \\ b &= g(a, b) \end{aligned}$$

We can also linearize about the equilibrium the usual way. We can also solve the linear system in the usual way- by looking at eigenvalues and eigenvectors:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

We can convert second order difference equations into a linear system (as we did with differential equations). Here's an example:

Convert the second order difference equation into a system of first order.

$$x_{n+2} + 3x_{n+1} + x_n = 0$$

SOLUTION: Let $a_n = x_n$ and $b_n = x_{n+1}$. Then:

$$\begin{aligned} a_{n+1} &= b_n \\ b_{n+1} &= -a_n - 3b_n \end{aligned}$$

Convert the following system into an equivalent second order difference equation:

$$\begin{aligned} a_{n+1} &= -a_n + b_n \\ b_{n+1} &= a_n - 3b_n \end{aligned}$$

SOLUTION: From the first equation, we have: $b_n = a_{n+1} + a_n$. Substitute this expression into the second equation to get:

$$a_{n+2} + a_{n+1} = a_n - 3(a_{n+1} + a_n) \quad \Rightarrow \quad a_{n+2} + 4a_{n+1} + 2a_n = 0$$

We could go ahead an solve this: The roots to the characteristic equation are $-2 \pm \sqrt{2}$, so the general solution is:

$$a_n = C_1(-2 + \sqrt{2})^n + C_2(-2 - \sqrt{2})^n$$

From this we get b_n:

$$b_n = a_{n+1} + a_n = C_1(-2 + \sqrt{2})^{n+1} + C_2(-2 - \sqrt{2})^{n+1} + C_1(-2 + \sqrt{2})^n + C_2(-2 - \sqrt{2})^n$$

Final notes

Finally, just as we had competing species and predator prey models as differential equations, we can also do almost exactly the same as difference equations.

We'll write them down and let you observe the similarities to the continuous cases:

$$\begin{aligned} a_{n+1} - a_n &= a_n(\epsilon_1 - \sigma_1 a_n) - \gamma_1 a_n b_n \\ b_{n+1} - b_n &= b_n(\epsilon_2 - \sigma_2 b_n) - \gamma_2 a_n b_n \end{aligned}$$

$$\begin{aligned} a_{n+1} - a_n &= a_n(\epsilon_1 - \sigma_1 a_n) - \gamma_1 a_n b_n \\ b_{n+1} - b_n &= -\epsilon_2 b_n + \gamma_2 a_n b_n \end{aligned}$$

Homework: Discrete Dynamical Systems

1. Solve:
   (a) $x_{n+1} = x_n + 1$
   (b) $x_{n+1} = 5x_n + n^2$
   (c) $x_{n+1} = \frac{1}{2}x_n + 3^n$

2. Assume the temperature of a roast in the oven increases at a rate proportional to the difference between the oven temperature (set to 400) and the roast temperature. If the roast enters the oven at 50 degrees, and is measured one hour later to be 90, when will the roast reach the FDA safe temperature of 160? (Hint: Write down, then solve the difference equation).

3. Convert the following system of difference equations to a second order difference equation, and solve it if $x_0 = y_0 = 1$.

$$\begin{aligned} x_{n+1} &= 2y_n \\ y_{n+1} &= 3x_n \end{aligned}$$

4. Solve the second order difference equation with $x_0 = 1$, $x_1 = -1$.
   (a) $x_{n+2} - x_n = 0$
   (b) $x_{n+2} + x_n = 0$
   (c) $x_{n+2} + 3x_{n+1} + x_n = 0$
   (d) $x_{n+2} = x_{n+1} + x_n$ (Do you recognize this famous difference equation? Typically we set $x_0 = 1, x_1 = 1$ in this example, so you can solve it that way.)

5. Consider the difference equation:

$$x_{n+2} + \alpha x_{n+1} + \beta x_n = 0$$

where we assume $\alpha^2 = 4\beta = 0$. Show that $x_n = n(-\alpha/2)^n$ is a solution.

6. Convert the following equations to equivalent systems of first order:

    (a) $x_{n+2} + x_{n+1} - x_n = 0$

    (b) $x_{n+2} + 3x_{n+1} + x_n = 0$

7. Convert the following systems of first order into an equivalent difference equation of second order.

    (a) $\begin{aligned} a_{n+1} &= a_n + b_n \\ b_{n+1} &= 3a_n + b_n \end{aligned}$

    (b) $\begin{aligned} a_{n+1} &= 2a_n + b_n \\ b_{n+1} &= a_n + 2b_n \end{aligned}$

8. Is it always possible to convert a system of two linear equations (as in the last problem) to a single second order difference equation? Explain.