## Review Solutions, Mathematical Modeling

1. Let $\mathbf{x}^{\prime}=\left[\begin{array}{rr}1 & 1 \\ 6 & -4\end{array}\right] \mathbf{x}$. Convert this system to an equivalent second order linear homogeneous differential equation, then solve that.
SOLUTION: If we use $x_{1}, x_{2}$ for the variables, we can use the first equation to solve for $x_{2}$ in terms of $x_{1}$, then substitute that into the second equation:

$$
\begin{aligned}
& x_{1}^{\prime}=x_{1}+x_{2} \\
& x_{2}^{\prime}=6 x_{1}-4 x_{2}
\end{aligned} \quad \Rightarrow \quad x_{2}=x_{1}^{\prime}-x_{1} \quad \Rightarrow \quad\left(x_{1}^{\prime}-x_{1}\right)^{\prime}=6 x_{1}-4\left(x_{1}^{\prime}-x_{1}\right)
$$

Simplifying this to get a second order equation with $x_{1}$ :

$$
x_{1}^{\prime \prime}-x_{1}^{\prime}=6 x_{1}-4 x_{1}^{\prime}+4 x_{1} \quad \Rightarrow \quad x^{\prime \prime}+3 x_{1}^{\prime}-10 x_{1}=0
$$

which gives us the characteristic equation: $r^{2}+3 r-10=0$, or $(r-2)(r+5)=0$. Therefore, the full solution for $x_{1}$ is given by:

$$
C_{1} \mathrm{e}^{2 t}+C_{2} \mathrm{e}^{-5 t}
$$

EXTRA NOTE: We can also solve this directly using eigenvalues and eigenvectors. Given the matrix $A$, the characteristic equation is the same:

$$
\lambda^{2}+3 \lambda-1=0 \quad \Rightarrow \quad \lambda=2,-5
$$

If $\lambda=2$, the eigenvector is found the usual way:

$$
(A-\lambda I) \mathbf{v}=0 \quad \Rightarrow \quad-v_{1}+v_{2}=0 \quad \Rightarrow \quad \mathbf{v}=\left[\begin{array}{l}
1 \\
1
\end{array}\right]
$$

Similarly, using $\lambda=-5$, the eigenvector is found by solving

$$
-4 v_{1}+v_{2}=0 \quad \Rightarrow \quad \mathbf{v}=\left[\begin{array}{l}
1 \\
4
\end{array}\right]
$$

The full solution to the system is therefore:

$$
\mathbf{x}(t)=C_{1} \mathrm{e}^{2 t}\left[\begin{array}{l}
1 \\
1
\end{array}\right]+C_{2} \mathrm{e}^{-5 t}\left[\begin{array}{l}
1 \\
4
\end{array}\right]=\left[\begin{array}{c}
C_{1} \mathrm{e}^{2 t}+C_{2} \mathrm{e}^{-5 t} \\
C_{1} \mathrm{e}^{2 t}+4 C_{2} \mathrm{e}^{-5 t}
\end{array}\right]
$$

Notice that the solution we got in the original question is the first line.
2. Let $y^{\prime \prime}-6 y^{\prime}+9 y=0$ with $y(0)=1, y^{\prime}(0)=2$. Convert this into an equivalent system of first order differential equations, then solve it using eigenvectors and eigenvalues.

SOLUTION: Let $x_{1}=y$ and $x_{2}=y^{\prime}$. Then the system of DEs we get:

$$
\begin{aligned}
& x_{1}^{\prime}=x_{2} \\
& x_{2}^{\prime}=-9 x_{1}+6 x_{2}
\end{aligned} \quad \Rightarrow \quad \mathbf{x}^{\prime}=\left[\begin{array}{rr}
0 & 1 \\
-9 & 6
\end{array}\right] \mathbf{x}
$$

With trace 6 and det 9 , the characteristic equation is $\lambda^{2}-6 \lambda+9=0$ (which is what we expected). Solving for $\lambda$, we get a double root of $\lambda=3$.
For the eigenvector:

$$
\left[\begin{array}{ll}
-3 & 1 \\
-9 & 3
\end{array}\right] \mathbf{v}=\overrightarrow{0} \quad \Rightarrow \quad \mathbf{v}=\left[\begin{array}{l}
1 \\
3
\end{array}\right]
$$

Since we have a double eigenvalue and only one eigenvector, we get the generalized eigenvector $\mathbf{w}$ that solves: $(A-\lambda I) \mathbf{w}=\mathbf{v}$ :

$$
\begin{array}{r}
-3 w_{1}+w_{2}=1 \\
-9 w_{1}+3 w_{2}=3
\end{array}
$$

Choose any vector that satisfies these equations. For example, $w_{1}=0$ and $w_{2}=1$.
Now the general solution is given by:

$$
\mathbf{x}(t)=\mathrm{e}^{3 t}\left(C_{1}\left[\begin{array}{l}
1 \\
3
\end{array}\right]+C_{2}\left(t\left[\begin{array}{l}
1 \\
3
\end{array}\right]+\left[\begin{array}{l}
0 \\
1
\end{array}\right]\right)\right)
$$

With the initial condition, you should find that $C_{1}=2 / 3$ and $C_{2}=1 / 3$.
EXTRA NOTE: If we had solved this directly, we would have gotten

$$
y(t)=\mathrm{e}^{3 t}\left(C_{1}+C_{2} t\right)
$$

which is what we get in the system as well. Notice that the second coordinate of the solution to the system is $y^{\prime}$ :

$$
y^{\prime}=3 C_{1} \mathrm{e}^{3 t}+3 C_{2} t \mathrm{e}^{3 t}+C_{2} \mathrm{e}^{3 t}=\mathrm{e}^{3 t}\left(3 C_{1}+3 t C_{2}+C_{2}\right)
$$

3. Given each matrix $A$ below, give the general solution to $\mathbf{x}^{\prime}=A \mathbf{x}$, and classify the equilibrium as to its stability (you may use the Poincaré Diagram, if needed).
(a) $\left[\begin{array}{rr}0 & 1 \\ -2 & -3\end{array}\right]$

SOLUTION:

$$
\mathbf{x}(t)=C_{1} \mathrm{e}^{-t}\left[\begin{array}{r}
1 \\
-1
\end{array}\right]+C_{2} \mathrm{e}^{-2 t}\left[\begin{array}{r}
-1 \\
2
\end{array}\right]
$$

(b) $\left[\begin{array}{rr}-4 & -17 \\ 2 & 2\end{array}\right]$

SOLUTIONS: The eigenvalues are $\lambda=-1 \pm 5 i$. Using $\lambda=-1+5 i$, we find the corresponding eigenvector:

$$
\left[\begin{array}{cc}
-4-(-1+5 i) & -17 \\
2 & 2-(-1+5 i)
\end{array}\right] \mathbf{v}=0
$$

Using the second equation, we get $\mathbf{v}=[-3+5 i, 2]^{T}$. To find the solution, we compute $\mathrm{e}^{\lambda t} \mathbf{v}$ :

$$
\mathrm{e}^{(-1+5 i) t}\left[\begin{array}{r}
-3+5 i \\
2
\end{array}\right]=\mathrm{e}^{-t}\left[\begin{array}{c}
(-3 \cos (5 t)-5 \sin (5 t))+i(-3 \sin (5 t)+5 \cos (5 t)) \\
2 \cos (5 t)+2 i \sin (5 t)
\end{array}\right]
$$

Therefore, the full solution is:

$$
\mathbf{x}(t)=\mathrm{e}^{-t}\left(C_{1}\left[\begin{array}{c}
-3 \cos (5 t)-5 \sin (5 t) \\
2 \cos (5 t)
\end{array}\right]+C_{2}\left[\begin{array}{c}
-3 \sin (5 t)+5 \cos (5 t) \\
2 \sin (5 t)
\end{array}\right]\right)
$$

(You might note that the origin here is a SPIRAL SINK)
(c) $\left[\begin{array}{ll}3 & -4 \\ 1 & -1\end{array}\right]$

SOLUTION: You should find a double eigenvalue, $\lambda=1,1$ with eigenvector $\mathbf{v}=$ $[2,1]^{T}$. We then need a generalized eigenvector $\mathbf{w}$ that satisfies the equation $(A-\lambda I) \mathbf{w}=\mathbf{v}$ :

$$
\left[\begin{array}{ll}
2 & -4 \\
1 & -2
\end{array}\right]\left[\begin{array}{l}
w_{1} \\
w_{2}
\end{array}\right]=\left[\begin{array}{l}
2 \\
1
\end{array}\right]
$$

Choose any $w_{1}, w_{2}$ that satisfies this relationship. For example, $w_{1}=1$ and $w_{2}=0$ is convenient. Now we write the solution:

$$
\mathbf{x}(t)=\mathrm{e}^{t}\left(C_{1}\left[\begin{array}{l}
2 \\
1
\end{array}\right]+C_{2}\left(t\left[\begin{array}{l}
2 \\
1
\end{array}\right]+\left[\begin{array}{l}
1 \\
0
\end{array}\right]\right)\right)
$$

4. Suppose we have brine pouring into tank $A$ at a rate of 2 gallons per minute, and salt is in the brine at a concentration of $1 / 2$ pound per gallon. Brine is being pumped into tank $A$ from tank $B$ (well mixed) at a rate of 1 gallon per minute. Brine is pumped out of tank $A$ at a rate of 3 gallons per minute to tank $B$, and brine is poured into tank $B$ from an external source at a rate of 2 gallons per minute, and $1 / 3$ pound of salt per gallon. Initially, both tanks have 100 gallons of clear water.
Write the system of differential equations that model the amount of salt in the tanks at time $t$.
TYPO, becoming part of the problem: Before solving, determine at what rate the well mixed solution needs to be pumped out of Tank $B$ to keep the tanks at 100 gallons of fluid for all time.
SOLUTION: We need to pump out 4 gallons per minute.
Now, we can write the differential equations. Recall that the model is "Rate in-Rate out". Let $A(t), B(t)$ be the amount (in pounds) of salt in tank $A, B$ respectively, at time $t$ in minutes. Then:

$$
\frac{d A}{d t}=\left(2 \cdot \frac{1}{2}+1 \cdot \frac{B}{100}\right)-3 \frac{A}{100}
$$

$$
\frac{d B}{d t}=\left(2 \cdot \frac{1}{3}+3 \frac{A}{100}\right)-5 \frac{B}{100}
$$

You could stop there, but let's put it in matrix form so it looks familiar:

$$
\left[\begin{array}{l}
A \\
B
\end{array}\right]^{\prime}=\frac{1}{100}\left[\begin{array}{rr}
-3 & 1 \\
3 & -5
\end{array}\right]\left[\begin{array}{l}
A \\
B
\end{array}\right]+\left[\begin{array}{r}
1 \\
2 / 3
\end{array}\right]
$$

5. Consider the system $\mathbf{x}^{\prime}=A \mathbf{x}+\mathbf{b}$ given below:

$$
\left[\begin{array}{l}
x_{1}^{\prime} \\
x_{2}^{\prime}
\end{array}\right]=\left[\begin{array}{ll}
1 & 3 \\
4 & 2
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]+\left[\begin{array}{l}
10 \\
10
\end{array}\right]
$$

(a) Find the equilibrium solution, $\mathbf{x}_{E}$.

SOLUTION: The equilibrium solution for a differential equation is where the derivative is zero.

$$
\left[\begin{array}{l}
0 \\
0
\end{array}\right]=\left[\begin{array}{ll}
1 & 3 \\
4 & 2
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]+\left[\begin{array}{l}
10 \\
10
\end{array}\right] \quad \mathbf{x}=\frac{1}{2-12}\left[\begin{array}{rr}
2 & -3 \\
-4 & 1
\end{array}\right]\left[\begin{array}{l}
-10 \\
-10
\end{array}\right]=\left[\begin{array}{l}
-1 \\
-3
\end{array}\right]
$$

(b) Show that, if $\mathbf{u}=\mathbf{x}-\mathbf{x}_{E}$, then the differential equation for $\mathbf{u}$ is: $\mathbf{u}^{\prime}=A \mathbf{u}$.

SOLUTION: We can show it in general-

$$
\mathbf{u}^{\prime}=\mathbf{x}^{\prime}-\mathbf{x}_{E}=A \mathbf{x}+\mathbf{b}-\mathbf{x}_{E}+A \mathbf{x}_{E}-A \mathbf{x}_{E}=A\left(\mathbf{x}-\mathbf{x}_{E}\right)+\mathbf{b}+A\left(-A^{-1} \mathbf{b}\right)=A\left(\mathbf{x}-\mathbf{x}_{E}\right)
$$

Therefore, $\mathbf{u}^{\prime}=A \mathbf{u}$.
(c) Solve the differential equation by first solving the DE for $\mathbf{u}$.

TYPO: The eigenvalues/eigenvectors for $A$ are not simple expressions, so write your answer symbolically, assuming two distinct eigenvalues. SOLUTION:

$$
\mathbf{u}(t)=C_{1} \mathrm{e}^{\lambda_{1} t} \mathbf{v}_{1}+C_{2} \mathrm{e}^{\lambda_{2} t} \mathbf{v}_{2} \Rightarrow \mathbf{x}(t)=C_{1} \mathrm{e}^{\lambda_{1} t} \mathbf{v}_{1}+C_{2} \mathrm{e}^{\lambda_{2} t} \mathbf{v}_{2}+\left[\begin{array}{l}
-1 \\
-3
\end{array}\right]
$$

6. Use the Poincaré Diagram to determine how the origin changes stability by changing $\alpha$ if

$$
\mathbf{x}^{\prime}=\left[\begin{array}{cc}
\alpha+1 & \alpha \\
2 & 1
\end{array}\right] \mathbf{x}
$$

SOLUTION: To use the Poincaré Diagram, we look at expressions for the trace, determinant and discriminant and determine where each is positive/negative/zero. In this situation,

$$
\operatorname{Tr}(A)=\alpha+2 \quad \operatorname{det}(A)=1-\alpha \quad \Delta=\alpha^{2}+8 \alpha
$$

Performing a sign chart analysis (at the bottom, divide the $\alpha$ number line by where each quantity is zero)

$$
\begin{array}{l|ccccc}
\alpha+2 & - & - & + & + & + \\
1-\alpha & + & + & + & + & - \\
\alpha(\alpha+8) & + & - & - & + & + \\
\hline & \alpha<-8 & -8<\alpha<-2 & -2<\alpha<0 & 0 \leq \alpha<1 & \alpha>1
\end{array}
$$

We can now read off the results, from left to right:

- If $\alpha<-8$, we have a sink.
- If $\alpha=-8$, we have a degenerate sink.
- If $-8<\alpha<-2$, we have a spiral sink.
- If $\alpha=-2$, we have a center.
- If $-2<\alpha<0$, we have a spiral source.
- If $\alpha=0$, we have a degenerate source.
- If $0<\alpha<1$, we have a source.
- If $\alpha=1$, we have a line of unstable fixed points.
- If $\alpha>1$, we have a saddle.

7. Let $F$ be given below, and linearize it at the given value.
(a) $\mathbf{F}(t)=\left[\begin{array}{r}t^{2}+3 t+2 \\ \sqrt{t+1}+1 \\ \sin (t)\end{array}\right] \quad$ at $\quad t=0$

SOLUTION: The "derivative" in this case is computed element-wise, so that:
$\mathbf{F}(0)=\left[\begin{array}{l}2 \\ 2 \\ 0\end{array}\right] \quad \mathbf{F}^{\prime}(0)=\left[\begin{array}{c}3 \\ 1 / 2 \\ 1\end{array}\right] \quad \Rightarrow \quad \mathbf{L}(t)=\mathbf{F}(0)+\mathbf{F}^{\prime}(0) t=\left[\begin{array}{l}2 \\ 2 \\ 0\end{array}\right]+t\left[\begin{array}{c}3 \\ 1 / 2 \\ 1\end{array}\right]$
(b) $f(x, y, z)=x^{2}+3 x+2 y+4 z-2 \quad$ at $\quad(x, y, z)=(1,-1,1)$

In this case, the linearization is given by:

$$
L(x, y, z)=f(1,-1,1)+f_{x}(1,-1,1)(x-1)+f_{y}(1,-1,1)(y+1)+f_{z}(1,-1,1)(z-1)
$$

Substituting everything in, we get:

$$
L(x, y, z)=4+5(x-1)+2(y+1)+4(z-1)
$$

(c) $\mathbf{F}(x, y)=\left[\begin{array}{c}x^{2}+3 x y-y+1 \\ y^{2}+2 x y+x^{2}-1\end{array}\right] \quad$ at $(x, y)=(1,0)$

In this case, the linearization is given by the following, if we think of $f(x, y)=$ $x^{2}+3 x y-y+1$ and $g(x, y)=y^{2}+2 x y+x^{2}-1$ :

$$
\mathbf{L}(x, y)=\mathbf{F}(1,0)+\left[\begin{array}{ll}
f_{x}(1,0) & f_{y}(1,0) \\
g_{x}(1,0) & g_{y}(1,0)
\end{array}\right]\left[\begin{array}{c}
x-1 \\
y
\end{array}\right]=\left[\begin{array}{l}
2 \\
0
\end{array}\right]+\left[\begin{array}{ll}
2 & 2 \\
2 & 2
\end{array}\right]\left[\begin{array}{c}
x-1 \\
y
\end{array}\right]
$$

8. For each nonlinear system below, perform a local linear analysis about all equilibria.
(a)

$$
\begin{aligned}
d x / d t & =x-x y \\
d y / d t & =y+2 x y
\end{aligned}
$$

We should find two equilbria. The origin is a source and the point $(-1 / 2,1)$ is a saddle.
(b)

$$
\begin{aligned}
d x / d t & =1+2 y \\
d y / d t & =1-3 x^{2}
\end{aligned}
$$

We should find two equilibria: $(-\sqrt{3},-1 / 2)$ and $(\sqrt{3},-1 / 2)$. The first equilibrium is saddle, the second is a center.
Side Remark: In this instance, the full nonlinear system actually has a spiral sink at the second equilibrium, but we would have failed to see it because of the linearization, as we discussed in class.
9. For each of the systems in question 8 , solve them by first computing $d y / d x$.
(a) In this case, we have a separable differential equation:

$$
\frac{d y}{d x}=\frac{y(1+2 x)}{x(1-y)} \Rightarrow \int \frac{1-y}{y} d y=\int \frac{1+2 x}{x} d x \quad \Rightarrow \quad \ln |y|-y=\ln |x|+2 x+C
$$

(b) In this case, we also have a separable differential equation:

$$
\frac{d y}{d x}=\frac{1-3 x^{2}}{1+2 y} \Rightarrow \int 1+2 y d y=\int 1-3 x^{2} d x \quad \Rightarrow \quad y+\frac{1}{2} y^{2}=x-x^{3}+C
$$

10. For 8(a) above, if $x$ and $y$ were two populations, what kinds of assumptions are being made to result in these differential equations?
SOLUTION: In the absence of the other, both populations experience exponential growth. In the presence of interactions between them, $x$ suffers and $y$ benefits (perhaps $y$ is eating $x!$ ).
11. Given $x^{\prime}=f(x, y)$ and $y^{\prime}=g(x, y)$, then a nullcline is a curve where $f$ or $g$ is 0 . Note that an equilibrium is where the nullclines intersect.

If $x^{\prime}=-4 x+y+x^{2}$ and $y^{\prime}=1-y$, then graph the nullclines, taking note of the equilbrium solutions. Is there an area in your drawing where $x^{\prime}<0$ and $y^{\prime}<0$ ? Make note of it.
SOLUTION: See the figure below. The region of interest is above the line and inside the parabola.

12. Is the following system an example of predator-prey or competing species? In either case, perform a local linear analysis:

$$
\begin{aligned}
& x^{\prime}=x(1-0.5 y) \\
& y^{\prime}=y(-0.75+0.25 x)
\end{aligned}
$$

SOLUTION: This is an example of predator-prey ( $x$ is the prey). There are two equilibria: $(0,0)$ and $(3,2)$. When we linearize about the origin, we get a saddle, and when we linearize about $(3,2)$, we get a center (which in fact does not persist in the full nonlinear case).
13. Solve:
(a) $x_{n+1}=x_{n}+1$

SOLUTION: You might try this one just from first principles:

$$
x_{0}, \quad x_{1}=x_{0}+1, \quad x_{2}=x_{0}+2, \cdots
$$

From which it is apparent that $x_{n}=x_{0}+n$, given any initial $x_{0}$.
(b) $x_{n+1}=5 x_{n}+n^{2}$

SOLUTION: The form of our solution is $c \cdot 5^{n}+p(n)$, where $p(n)$ is a full parabola: $A n^{2}+B n+c$. We substitute $p(n)$ into the difference equation to get equations for $A, B, C$ :

$$
A\left(n^{2}+2 n+1\right)+B(n+1)+C=5 A n^{2}+5 B n+5 C+n^{2}
$$

This must hold for each $n$, so therefore we get three equations (one for $n^{2}$ terms, one for $n$ terms, one for the constants):

$$
\begin{gathered}
A=5 A+1 \quad \Rightarrow \quad A=-\frac{1}{4} \\
2 A+B=5 B \quad \Rightarrow \quad 4 B=2 A \quad \Rightarrow \quad B=\frac{1}{2} A=-\frac{1}{8} \\
A+B+C=5 C \quad \Rightarrow \quad C=\frac{1}{4}\left(-\frac{1}{4}-\frac{1}{8}\right)=-\frac{3}{32}
\end{gathered}
$$

Therefore, the full solution is:

$$
x_{n}=c \cdot 5^{n}-\frac{1}{4} n^{2}-\frac{1}{8} n-\frac{3}{32}
$$

(c) $x_{n+1}=\frac{1}{2} x_{n}+3^{n}$

SOLUTION: Try going at this from first principles.

$$
\begin{gathered}
x_{1}=\frac{1}{2} x_{0}+1 \\
x_{2}=\frac{1}{2} x_{1}+3^{1}=\frac{1}{2^{2}} x_{0}+\frac{1}{2}+3 \\
x_{3}=\frac{1}{2^{3}} x_{0}+\frac{1}{2^{2}}+\frac{3}{2}+3^{2}, \text { etc. }
\end{gathered}
$$

Did you get the following pattern?

$$
x_{n}=\frac{1}{2^{n}} x_{0}+\frac{1}{2^{n-1}}+\frac{3}{2^{n-2}}+\cdots+\frac{3^{n-2}}{2}+3^{n}
$$

If you got this far, great! It is possible to show that the solution can also be written as:

$$
x_{n}=c \cdot\left(\frac{1}{2}\right)^{n}+\frac{2}{5} \cdot 3^{n}
$$

where $c=x_{0}-2 / 5$ if we use some geometric sum formulas. It's fine if you got the other version, though.
14. Assume the temperature of a roast in the oven increases at a rate proportional to the difference between the oven temperature (set to 400) and the roast temperature. If the roast enters the oven at 50 degrees, and is measured one hour later to be 90, when will the roast reach the FDA safe temperature of 160 ? (Hint: Write down, then solve the difference equation).
SOLUTION: From what is given, we have the following, where $R_{n}$ is the temperature of the roast at hour $n$, and $k$ is our constant of proportionality:

$$
R_{n+1}-R_{n}=k\left(400-R_{n}\right)
$$

From the measurements we're given, we can solve for $k\left(R_{0}=50\right.$ and $\left.R_{1}=90\right)$ :

$$
90-50=k(400-50) \quad \Rightarrow \quad k=\frac{4}{35}
$$

Our difference equation is now:

$$
R_{n+1}=(1-k) R_{n}+400 k=\frac{31}{35} R_{n}+\frac{320}{7}
$$

Now, given that $R_{n+1}=a R_{n}+b$, we can solve this directly. Recall that, if we define $R_{E}$ as the equilibrium solution, then:

$$
R_{n}=a^{n}\left(R_{0}-R_{E}\right)+R_{E}
$$

If we compute the equilibrium, we get $R_{E}=400$ (which makes sense). Now:

$$
R_{n}=(-350) \cdot\left(\frac{31}{35}\right)^{n}+400
$$

Finally, setting $R_{n}=160$, we see that $n \approx 3.1$ hours.
NOTE: I won't have you do a lot of numerical work like this without a calculator (and you won't have a calculator). However, you should have been able to set up the difference equation and solve for the growth constant $k$.
15. Convert the following system of difference equations to a second order difference equation, and solve it if $x_{0}=y_{0}=1$.

$$
\begin{aligned}
x_{n+1} & =2 y_{n} \\
y_{n+1} & =3 x_{n}
\end{aligned}
$$

SOLUTION: Sorry about the slightly messy numbers. Here, we get $x_{n+2}=6 x_{n}$, so with the ansatz that $x_{n}=\lambda^{n}$, the characteristic equation is:

$$
\lambda^{2}-6=0 \quad \Rightarrow \quad \lambda= \pm \sqrt{6}
$$

so the general solution is given by

$$
x_{n}=C_{1}(\sqrt{6})^{n}+C_{2}(-\sqrt{6})^{n}
$$

And, given that $x_{0}=1$ and $x_{1}=2 y_{0}=2$, then we get that

$$
C_{1,2}=\frac{3 \pm \sqrt{6}}{6}
$$

16. Solve the second order difference equation with $x_{0}=1, x_{1}=-1$.
(a) $x_{n+2}-x_{n}=0$

SOLUTION: The characteristic equation: $\lambda^{2}-1=0$, so $\lambda= \pm 1$ and the general solution is:

$$
x_{n}=C_{1}(1)^{n}+C_{2}(-1)^{n}
$$

We get $C_{1}=0$ and $C_{2}=1$, so the solution is: $x_{n}=(-1)^{n}$.
(b) $x_{n+2}+x_{n}=0$

SOLUTION: In this case, we get complex roots to the characteristic equation. Writing $\lambda=r \mathrm{e}^{i \theta}$, then the general solution is

$$
x_{n}=r^{n}\left(C_{1} \cos (n \theta)+C_{2} \sin (n \theta)\right)
$$

In this case, $\lambda= \pm i$, so that $r=1$ and $\theta=\frac{\pi}{2}$ :

$$
x_{n}=C_{1} \cos \left(\frac{n \pi}{2}\right)+C_{2} \sin \left(\frac{n \pi}{2}\right)
$$

Solving for $C_{1}, C_{2}$, we should get $C_{1}=1$ and $C_{2}=-1$.
(c) $x_{n+2}+3 x_{n+1}+x_{n}=0$

SOLUTION: TYPO: As is, the solution is a bit messy. Change the 3 to a 2 so that the characteristic equation becomes $\lambda^{2}+2 \lambda+1=(\lambda+1)^{2}=0$
Now we have a repeated root of $\lambda=-1$, and the general solution is:

$$
x_{n}=(-1)^{n}\left(C_{1}+n C_{2}\right)
$$

Solving for the constants, $C_{1}=1$ and $C_{2}=0$.
(d) $x_{n+2}=x_{n+1}+x_{n}$ (Do you recognize this famous difference equation? Typically we set $x_{0}=1, x_{1}=1$ in this example, so you can solve it that way.)
SOLUTION: OK, this one is messy, but it's supposed to be that way! The characteristic equation gives us

$$
\lambda_{1,2}=\frac{1 \pm \sqrt{5}}{2}
$$

so that

$$
x_{n}=C_{1} \lambda_{1}^{n}+C_{2} \lambda_{2}^{n}
$$

17. Consider the difference equation:

$$
x_{n+2}+\alpha x_{n+1}+\beta x_{n}=0
$$

where we assume $\alpha^{2}=4 \beta=0$. Show that $x_{n}=n(-\alpha / 2)^{n}$ is a solution.
SOLUTION: Substitute the expression in, and show that you have a true statement. Substituting, we get:

$$
(n+2)\left(-\frac{\alpha}{2}\right)^{n+2}+\alpha(n+1)\left(-\frac{\alpha}{2}\right)^{n+1}+\beta n\left(-\frac{\alpha}{2}\right)^{n}=0
$$

Factor out $(-\alpha / 2)^{n}$ (and divide it out), then multiply by 4 :

$$
\begin{gathered}
(n+2)\left(-\frac{\alpha}{2}\right)^{2}+\alpha(n+1)\left(-\frac{\alpha}{2}\right)+\beta n=0 \\
\alpha^{2}(n+2)-2 \alpha^{2}(n+1)+4 \beta n=0
\end{gathered}
$$

We need to show that this is true for all $n$, which means the coefficient for $n$ needs to sum to zero, as does the constant term:

$$
\begin{array}{lr}
\mathrm{n} \text { terms } & \alpha^{2}-2 \alpha^{2}+4 \beta=0 \\
\text { constants } & 2 \alpha^{2}-2 \alpha^{2}=0
\end{array}
$$

For the first equation, we have $\alpha^{2}=4 \beta$, so that is true. The second equation is always true as well. Therefore, the expression is a solution to the difference equation.
18. Convert the following equations to equivalent systems of first order:
(a) $x_{n+2}+x_{n+1}-x_{n}=0$

SOLUTION: If we let $u_{n}=x_{n}$ and $v_{n}=x_{n+1}$, then the system is

$$
\begin{array}{r}
u_{n+1}=v_{n} \\
v_{n+1}=u_{n}-v_{n}
\end{array}
$$

(b) $x_{n+2}+3 x_{n+1}+x_{n}=0$

SOLUTION: If we let $u_{n}=x_{n}$ and $v_{n}=x_{n+1}$, then the system is

$$
\begin{array}{r}
u_{n+1}=v_{n} \\
v_{n+1}=-u_{n}-3 v_{n}
\end{array}
$$

19. Convert the following systems of first order into an equivalent difference equation of second order.
(a) $\begin{aligned} a_{n+1} & =a_{n}+b_{n} \\ b_{n+1} & =3 a_{n}+b_{n}\end{aligned}$

SOLUTION: Let $b_{n}=a_{n+1}-a_{n}$ from the first equation, then subsitute into the second:

$$
a_{n+2}-a_{n+1}=3 a_{n}+a_{n+1}-a_{n} \quad \Rightarrow \quad a_{n+2}-2 a_{n+1}-2 a_{n}=0
$$

(b) $\begin{aligned} & a_{n+1}=2 a_{n}+b_{n} \\ & b_{n+1}=a_{n}+2 b_{n}\end{aligned}$

SOLUTION: Let $b_{n}=a_{n+1}-2 a_{n}$ from the first equation, then subsitute into the second:

$$
a_{n+2}-2 a_{n+1}=a_{n}+2 a_{n+1}-2 a_{n} \quad \Rightarrow \quad a_{n+2}-4 a_{n+1}+a_{n}=0
$$

20. Is it always possible to convert a system of two linear equations (as in the last problem) to a single second order difference equation? Explain.
SOLUTION: It is possible if you can write one of the variables in terms of the other. That is not possible in a system like:

$$
x^{\prime}=a x, \quad y^{\prime}=b y
$$

