The SIR Model for the Spread of Disease[1]

We first identify our variables- Let t be time, measured in days. We will track three groups of people- Susceptible, Infected, and Recovered.

We will let $S(t), I(t)$, and $R(t)$ be these populations at time t. For an initial simplification, we will assume no change in the total population, N.

Some of the computations will be easier to make if we take the variables to be percentages rather than the actual counts, so let lowercase values indicate percents:

$$s(t) = S(t)/N, \quad i(t) = I(t)/N, \quad r(t) = R(t)/N$$

Questions

1. Beginning with non-zero numbers in s, i, r, what should the graphs of these functions look like? (Try sketching)

2. Explain why, at each time t, $s(t) + i(t) + r(t) = 1$.

Assumptions

No one is added to the susceptible group, since we are ignoring births and immigration. The only way an individual leaves the susceptible group is by becoming infected. We assume that the time-rate of change of $S(t)$, the number of susceptibles, depends on the number already susceptible, the number of individuals already infected, and the amount of contact between susceptibles and infecteds.

In particular, suppose that each infected individual has a fixed number b of contacts per day that are sufficient to spread the disease. Not all these contacts are with susceptible individuals. If we assume a homogeneous mixing of the population, the fraction of these contacts that are with susceptibles is $s(t)$. Thus, on average, each infected individual generates $bs(t)$ new infected individuals per day.

We also assume that a fixed fraction k of the infected will recover during any given day. For example, if the average duration of infection is three days, then on average, one third of the currently infected population recovers each day.[2]

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[1] Modified from Smith and Moore on the MAA website.

[2] Strictly speaking, what we mean by "infected" is really "infectious," that is, capable of spreading the disease to a susceptible person. A "recovered" person can still feel miserable, and might even die later from pneumonia.

The Equations

1. For the change in the susceptible population, explain carefully each component of the DE:
$$\frac{dS}{dt} = -bs(t)I(t)$$
In particular, note $I(t)$ and the negative sign.

2. Explain how to arrive at the DE: $s' = -bs(t)i(t)$

3. For the recovered population, explaing how we get: $r' = ki(t)$.

4. For the infected, first explain: $s' + i' + r' = 0$, so that:
$$i' = bs(t)i(t) - ki(t)$$

The SIR model

$$\begin{aligned} s' &= -bs(t)i(t) \\ i' &= bs(t)i(t) - ki(t) \\ r' &= ki(t) \end{aligned}$$

Taking an example, suppose we start with 7.9 million people, 10 infected. We will need estimates of k and b- If we assume an average of three days for infected, and estimate that each infected would make an infecting contact every two days, then $k = 1/3$ and $b = 1/2$. See the resulting solution in Maple, plotted over 140 days.