Exam I Review

Modeling II, Spring 2004

1. Determine the projection matrix that takes a vector $\mathbf{x}$ and outputs the projection of $\mathbf{x}$ onto the plane whose normal vector is $[1, 1, 1]^T$.

2. Find (by hand) the eigenvectors and eigenvalues of the matrix A:

$$A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}, \quad A \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

3. (Referring to the previous exercise) We could've predicted that the eigenvalues of the second matrix would be real, and that the eigenvectors would be orthogonal. Why?

4. Compute the evals and evecs of $A^T A$ if

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}$$

5. Compute the orthogonal projector to the span of $\mathbf{x}$, if $\mathbf{x} = [1, 1, 1]^T$.

6. Show that $\dfrac{\mathbf{x}\mathbf{x}^T}{\|\mathbf{x}\|^2}$ is a projector, and is an orthogonal projector.

7. Let

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

   Find $[\mathbf{x}]_U$. Find the projection of $\mathbf{x}$ into the subspace spanned by the columns of U. Find the distance between $\mathbf{x}$ and the subspace spanned by the columns of U.

8. Show that $\text{Null}(A) \perp \text{Row}(A)$.

9. Show that, if X is invertible, then $X^{-1}AX$ and A have the same eigenvalues.

10. How do we "double-center" a matrix of data?

11. True or False, and give a short reason:

    (a) An orthogonal matrix is an orthogonal projector.

    (b) If the rank of A is 3, the dimension of the row space is 3.

    (c) If the correlation coefficient between two sets of data is 1, then the data sets are the same.

(d) If the correlation coefficient between two sets of data is 0, then there is no functional relationship between the two sets of data.

(e) If U is a 4×2 matrix, then $U^T U = I$.

(f) If U is a 4×2 matrix, then $UU^T = I$.

(g) If A is not invertible, then $\lambda = 0$ is an eigenvalue of A.

(h) Let

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$

Then the rank of AA^T is 2.

12. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be the eigenvectors of $A^T A$, where A is $m \times n$.

(a) True or false? The eigenvectors form an orthogonal basis of $\mathbb{R}^n$.

(b) Show that, if $\mathbf{x} \in \mathbb{R}^n$, then the i^{th} coordinate of $\mathbf{x}$ is $\mathbf{x}^T \mathbf{v}_i$.

(c) Let $\alpha_1, \ldots, \alpha_n$ be the coordinates of $\mathbf{x}$ with respect to $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Show that

$$\|\mathbf{x}\|_2 = \alpha_1^2 + \alpha_2^2 + \ldots + \alpha_n^2$$

(d) Show that $A\mathbf{v}_i \perp A\mathbf{v}_j$

(e) Show that $A\mathbf{v}_i$ is an eigenvector of AA^T.