Exam I Review Solutions to Problem 12

12. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be the eigenvectors of $A^T A$, where A is $m \times n$.

   (a) True or false? The eigenvectors form an orthogonal basis of $\mathbb{R}^n$.
   True. Since $A^T A$ is symmetric, the spectral theorem says that the eigenvectors form an orthogonal basis for $\mathbb{R}^n$.

   (b) Show that, if $\mathbf{x} \in \mathbb{R}^n$, then the i^{th} coordinate of $\mathbf{x}$ is $\mathbf{x}^T \mathbf{v}_i$.
   Since the $\mathbf{v}_i$'s form a basis,

$$\mathbf{x} = \alpha_1 \mathbf{v}_1 + \ldots + \alpha_n \mathbf{v}_n$$

   Now take the dot product of both sides with $\mathbf{v}_i$:

$$\mathbf{v}_i^T \mathbf{x} = \alpha_1 \mathbf{v}_j^T \mathbf{v}_1 + \ldots + \alpha_i \mathbf{v}_i^T \mathbf{v}_i + \ldots \alpha_n \mathbf{v}_i^T \mathbf{v}_n = 0 + 0 + \alpha_i + 0 + 0$$

   assuming that $\|\mathbf{v}_i\| = 1$.

   (c) Let $\alpha_1, \ldots, \alpha_n$ be the coordinates of $\mathbf{x}$ with respect to $\mathbf{v}_1, \ldots, \mathbf{v}_n$.
   Show that

$$\|\mathbf{x}\|_2 = \alpha_1^2 + \alpha_2^2 + \ldots + \alpha_n^2$$

   We can take:

$$\mathbf{x} = \alpha_1 \mathbf{v}_1 + \ldots + \alpha_n \mathbf{v}_n$$

   so that $\|\mathbf{x}\|^2 = \mathbf{x}^T \mathbf{x}$ which expands to:

$$(\alpha_1 \mathbf{v}_1 + \ldots + \alpha_n \mathbf{v}_n)^T (\alpha_1 \mathbf{v}_1 + \ldots + \alpha_n \mathbf{v}_n)$$

   The transpose of a sum is the sum of the transposes:

$$\left(\alpha_1 \mathbf{v}_1^T + \ldots + \alpha_n \mathbf{v}_n^T\right) (\alpha_1 \mathbf{v}_1 + \ldots + \alpha_n \mathbf{v}_n)$$

   and expanding this product we get:

$$\alpha_1^2 \mathbf{v}_1^T \mathbf{v}_1 + \alpha_2^2 \mathbf{v}_2^T \mathbf{v}_2 + \ldots + \alpha_n^2 \mathbf{v}_n^T \mathbf{v}_n + 0 + 0 + 0 + \ldots$$

   Assuming that the basis has been normalized,

$$\|\mathbf{x}\|^2 = \sum_{j=1}^{n} \alpha_j^2$$

   Alternatively, by the Pythagorean Theorem,

$$\|\alpha_1 \mathbf{v}_1 + \ldots + \alpha_n \mathbf{v}_n\|^2 = \|\alpha_1 \mathbf{v}_1\|^2 + \|\alpha_2 \mathbf{v}_2\|^2 + \ldots + \|\alpha_n \mathbf{v}_n\|^2 = \sum_{j=1}^{n} \alpha_j^2$$

(d) Show that $A\mathbf{v}_i \perp A\mathbf{v}_j$

We are given that $\mathbf{v}_i \perp \mathbf{v}_j$, since A^TA is symmetric. We want to show that:

$$(A\mathbf{v}_i) \cdot A\mathbf{v}_j = 0$$

Proof:

$$(A\mathbf{v}_i) \cdot A\mathbf{v}_j = (A\mathbf{v}_i)^T A\mathbf{v}_j = \left(\mathbf{v}_i^T A^T\right) A\mathbf{v}_j = \mathbf{v}_i^T \left(A^T A\mathbf{v}_j\right) = \lambda_j \mathbf{v}_i^T \mathbf{v}_j = 0$$

(e) Show that $A\mathbf{v}_i$ is an eigenvector of AA^T.

We want to show that:

$$AA^T(A\mathbf{v}_i) = \lambda A\mathbf{v}_i$$

for some λ.

Proof:

$$AA^T(A\mathbf{v}_i) = A(A^TA)\mathbf{v}_i = \lambda_i A\mathbf{v}_i$$