Review Questions, Mathematical Modeling

1. Let $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 6 & -4 \end{bmatrix} \mathbf{x}$. Convert this system to an equivalent second order linear homogeneous differential equation, then solve that analytically.

2. Let $y'' - 6y' + 9y = 0$ with $y(0) = 1$, $y'(0) = 2$. Convert this into an equivalent system of first order differential equations, then solve it analytically using eigenvectors and eigenvalues.

3. Given each matrix A below, give the general solution to $\mathbf{x}' = A\mathbf{x}$, and classify the equilibrium as to its stability (you may use the Poincaré Diagram, if needed).

   (a) $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} -4 & -17 \\ 2 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

4. Suppose we have brine pouring into tank A at a rate of 2 gallons per minute, and salt is in the brine at a concentration of 1/2 pound per gallon. Brine is being pumped into tank A from tank B (well mixed) at a rate of 1 gallon per minute. Brine is pumped out of tank A at a rate of 3 gallons per minute to tank B, and brine is poured into tank B from an external source at a rate of 2 gallons per minute, and 1/3 pound of salt per gallon. Initially, both tanks have 100 gallons of clear water.

   Write the system of differential equations that model the amount of salt in the tanks at time t.

5. Consider the system $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$ given below:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

   (a) Find the equilibrium solution, $\mathbf{x}_E$.

   (b) Show that, if $\mathbf{u} = \mathbf{x} - \mathbf{x}_E$, then the differential equation for $\mathbf{u}$ is: $\mathbf{u}' = A\mathbf{u}$.

   (c) Solve the differential equation by first solving the DE for $\mathbf{u}$.

6. Use the Poincaré Diagram to determine how the origin changes stability by changing α if

$$\mathbf{x}' = \begin{bmatrix} \alpha + 1 & \alpha \\ 2 & 1 \end{bmatrix} \mathbf{x}$$

   (Hint: Consider how the trace, determinant and discriminant change with α)

7. Let F be given below, and linearize it at the given value.

   (a) $\mathbf{F}(t) = \begin{bmatrix} t^2 + 3t + 2 \\ \sqrt{t+1} + 1 \\ \sin(t) \end{bmatrix}$ at $t = 0$

   (Hint: Linearization is done component-wise in this case).

(b) $f(x,y,z) = x^2 + 3x + 2y + 4z - 2 \qquad$ at $\qquad (x,y,z) = (1,-1,1)$

(c) $\mathbf{F}(x,y) = \begin{bmatrix} x^2 + 3xy - y + 1 \\ y^2 + 2xy + x^2 - 1 \end{bmatrix}$ at $(x,y) = (1,0)$

8. For each nonlinear system below, perform a local linear analysis about all equilibria.

(a) $\begin{aligned} dx/dt &= x - xy \\ dy/dt &= y + 2xy \end{aligned}$ (b) $\begin{aligned} dx/dt &= 1 + 2y \\ dy/dt &= 1 - 3x^2 \end{aligned}$

9. For each of the systems in question 8, solve them by first computing dy/dx.

10. For 8(a) above, if x and y were two populations, what kinds of assumptions are being made to result in these differential equations?

11. Given $x' = f(x,y)$ and $y' = g(x,y)$, then a **nullcline** is a curve where f or g is 0. Note that an equilibrium is where the nullclines intersect.

    If $x' = -4x + y + x^2$ and $y' = 1 - y$, then graph the nullclines, taking note of the equilbrium solutions. Is there an area in your drawing where $x' < 0$ and $y' < 0$? Make note of it.

12. Is the following system an example of predator-prey or competing species? In either case, perform a local linear analysis:

$$\begin{aligned} x' &= x(1 - 0.5y) \\ y' &= y(-0.75 + 0.25x) \end{aligned}$$

13. Assume the temperature of a roast in the oven increases at a rate proportional to the difference between the oven temperature (set to 400) and the roast temperature. If the roast enters the oven at 50 degrees, and is measured one hour later to be 90, when will the roast reach the FDA safe temperature of 160? (Hint: Write down, then solve the difference equation).

14. Consider the sytem of differential equations below.

$$\frac{dx}{dt} = x(1 - x - y), \qquad \frac{dy}{dt} = y\left(\alpha - y - \frac{1}{2}x\right)$$

where you may assume that $x \geq 0, y \geq 0$, and $\alpha \geq 0$.

(a) Draw the nullclines, and locate the equilbria graphically. You might note that $\alpha = 1$ and $\alpha = 1/2$ are some special cases to consider.

(b) Linearize the system.

(c) Analyze what happens at the origin in terms of α using the Poincare Diagram.

(d) If $\alpha = \frac{3}{4}$, linearize at $(1/2, 1/2)$ and give the results.

15. Consider the IVP:

$$\begin{aligned}\frac{dx}{dt} &= -x + 3z \\ \frac{dy}{dt} &= -y + 2z \\ \frac{dz}{dt} &= x^2 - 2z\end{aligned}$$

where $x(0) = 0, y(0) = 1/2$, and $z(0) = 3$. Further, we want the solution for $0 \le t \le 1.5$

(a) Numerically solve the system of equations given above, and print/save the file you used for the derivatives. First use Euler's method (forward), with step size 0.1, and plot the result.

(b) Repeat the previous solution, but use our own Runge-Kutta algorithm from class with step size 0.1. Plot the result.

(c) Repeat the solution, but use Octave's built-in function `ode23`, and plot the result.

(d) Linearize the system at the origin, then use Octave to find the eigenvalues. Considering the output, is the origin a sink, a source, or something else?

Notes about plotting in three dimensions

If you haven't done so already, before we start, go to Octave's menu on the upper right side of the browser, then find the check box next to `Render new plots inline`, and be sure that is unchecked. That will allow you to download your plots.

The command for a three dimensional plot is:

```
plot3(x,y,z)
```

where x, y, z are vectors of the same length and represent the x, y, z coordinates of the points being plotted. The vectors may be three row vectors or three column vectors (as long as you are consistent, it doesn't matter which).

For example, if the output to my ODE solver is a matrix y that is 200×3 (so that points are rows, like in `ode23`), then the command to plot the solution would be

```
plot3(y(:,1),y(:,2),y(:,3),'ko-');
```

where the optional argument `ko-` will plot the points using black circles and then connect the dots. If our matrix y is instead 3×200 (so that the points are in columns, as in `euler_forward`), then the command would be

```
plot3(y(1,:),y(2,:),y(3,:),'k*-');
```

Now the optional argument indicates that Octave will plot the points using asterisks and will connect the dots.