Review Solutions, Mathematical Modeling

1. Let $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 6 & -4 \end{bmatrix} \mathbf{x}$. Convert this system to an equivalent second order linear homogeneous differential equation, then solve that.

   SOLUTION: If we use x_1, x_2 for the variables, we can use the first equation to solve for x_2 in terms of x_1, then substitute that into the second equation:

$$\begin{array}{ll} x_1' & = x_1 + x_2 \\ x_2' & = 6x_1 - 4x_2 \end{array} \quad \Rightarrow \quad x_2 = x_1' - x_1 \quad \Rightarrow \quad (x_1' - x_1)' = 6x_1 - 4(x_1' - x_1)$$

   Simplifying this to get a second order equation with x_1:

$$x_1'' - x_1' = 6x_1 - 4x_1' + 4x_1 \quad \Rightarrow \quad x'' + 3x_1' - 10x_1 = 0$$

   Solving the sytem using eigenvalues and eigenvectors:

   Given the matrix A, the characteristic equation is the same:

$$\lambda^2 + 3\lambda - 10 = 0 \quad \Rightarrow \quad \lambda = 2, -5$$

   If $\lambda = 2$, the eigenvector is found the usual way:

$$(A - \lambda I)\mathbf{v} = 0 \quad \Rightarrow \quad -v_1 + v_2 = 0 \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

   Similarly, using $\lambda = -5$, the eigenvector is found by solving

$$6v_1 + v_2 = 0 \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

   The full solution to the system is therefore:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \mathrm{e}^{-5t} \begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} C_1 \mathrm{e}^{2t} + C_2 \mathrm{e}^{-5t} \\ C_1 \mathrm{e}^{2t} + 4C_2 \mathrm{e}^{-5t} \end{bmatrix}$$

   Finally, in this case, the origin (equilibrium solution) is classified as a SADDLE.

2. Let $y'' - 6y' + 9y = 0$ with $y(0) = 1$, $y'(0) = 2$. Convert this into an equivalent system of first order differential equations, then solve it using eigenvectors and eigenvalues.

   SOLUTION: Let $x_1 = y$ and $x_2 = y'$. Then the system of DEs we get:

$$\begin{array}{ll} x_1' & = x_2 \\ x_2' & = -9x_1 + 6x_2 \end{array} \quad \Rightarrow \quad \mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix} \mathbf{x}$$

   With trace 6 and det 9, the characteristic equation is $\lambda^2 - 6\lambda + 9 = 0$ (which is what we expected). Solving for λ, we get a double root of $\lambda = 3$.

We don't need the eigenvector, but we do need that vector $\mathbf{w}$. The initial condition is specified: $x_1(0) = 1$, and $x_2(0) = 2$. Then:

$$\begin{array}{ll} (a-\lambda)x_0 + by_0 & = w_1 \\ cx_0 + (d-\lambda)y_0 & = w_2 \end{array} \quad \Rightarrow \quad \begin{array}{ll} (0-3)\cdot 1 + 1\cdot 2 & = w_1 \\ -9\cdot 1 + (6-3)\cdot 2 & = w_2 \end{array} \quad \Rightarrow \quad \mathbf{w} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

Now the general solution is given by:

$$\mathbf{x}(t) = \mathrm{e}^{3t}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} -1 \\ -3 \end{bmatrix}\right)$$

3. Given each matrix A below, give the general solution to $\mathbf{x}' = A\mathbf{x}$, and classify the equilibrium as to its stability (you may use the Poincaré Diagram, if needed).

   (a) $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$
   Summary of the SOLUTION: The origin is a SINK, and the solution is:

$$\mathbf{x}(t) = C_1\mathrm{e}^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2\mathrm{e}^{-2t}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

   (b) $\begin{bmatrix} -4 & -17 \\ 2 & 2 \end{bmatrix}$
   SOLUTIONS: The eigenvalues are $\lambda = -1 \pm 5i$. Using $\lambda = -1 + 5i$, we find the corresponding eigenvector:

$$\begin{bmatrix} -4-(-1+5i) & -17 \\ 2 & 2-(-1+5i) \end{bmatrix}\mathbf{v} = 0$$

   Using the second equation, we get $\mathbf{v} = [-3+5i, 2]^T$. To find the solution, we compute $\mathrm{e}^{\lambda t}\mathbf{v}$:

$$\mathrm{e}^{(-1+5i)t}\begin{bmatrix} -3+5i \\ 2 \end{bmatrix} = \mathrm{e}^{-t}\begin{bmatrix} (-3\cos(5t) - 5\sin(5t)) + i(-3\sin(5t) + 5\cos(5t)) \\ 2\cos(5t) + 2i\sin(5t) \end{bmatrix}$$

   Therefore, the full solution is:

$$\mathbf{x}(t) = \mathrm{e}^{-t}\left(C_1\begin{bmatrix} -3\cos(5t) - 5\sin(5t) \\ 2\cos(5t) \end{bmatrix} + C_2\begin{bmatrix} -3\sin(5t) + 5\cos(5t) \\ 2\sin(5t) \end{bmatrix}\right)$$

   The origin here is a SPIRAL SINK.

(c) $\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

SOLUTION: You should find a double eigenvalue, $\lambda = 1, 1$ with eigenvector $\mathbf{v} = [2, 1]^T$. With no initial condition, assume it to be $x_1(0) = x_0$ and $x_2(0) = v_0$, so that we can compute $\mathbf{w}$ as:

$$\begin{aligned} (3-1)\cdot x_0 - 4v_0 &= w_1 \\ 1\cdot x_0 + (-1-1)v_0 &= w_2 \end{aligned} \qquad \Rightarrow \qquad \mathbf{w} = \begin{bmatrix} 2x_0 - 4v_0 \\ x_0 - 2v_0 \end{bmatrix}$$

Now we write the solution:

$$\mathbf{x}(t) = \mathrm{e}^t \left(\begin{bmatrix} x_0 \\ v_0 \end{bmatrix} + t \begin{bmatrix} 2x_0 - 4v_0 \\ x_0 - 2v_0 \end{bmatrix} \right)$$

4. Suppose we have brine pouring into tank A at a rate of 2 gallons per minute, and salt is in the brine at a concentration of 1/2 pound per gallon. Brine is being pumped into tank A from tank B (well mixed) at a rate of 1 gallon per minute. Brine is pumped out of tank A at a rate of 3 gallons per minute to tank B, and brine is poured into tank B from an external source at a rate of 2 gallons per minute, and 1/3 pound of salt per gallon. Initially, both tanks have 100 gallons of clear water.

   Write the system of differential equations that model the amount of salt in the tanks at time t.

   TYPO, becoming part of the problem: Before solving, determine at what rate the well mixed solution needs to be pumped out of Tank B to keep the tanks at 100 gallons of fluid for all time.

   SOLUTION: We need to pump out 4 gallons per minute.

   Now, we can write the differential equations. Recall that the model is "Rate in-Rate out". Let $A(t), B(t)$ be the amount (in pounds) of salt in tank A, B respectively, at time t in minutes. Then:

$$\frac{dA}{dt} = \left(2\cdot\frac{1}{2} + 1\cdot\frac{B}{100}\right) - 3\frac{A}{100}$$

$$\frac{dB}{dt} = \left(2\cdot\frac{1}{3} + 3\frac{A}{100}\right) - 5\frac{B}{100}$$

   You could stop there, but let's put it in matrix form so it looks familiar:

$$\begin{bmatrix} A \\ B \end{bmatrix}' = \frac{1}{100}\begin{bmatrix} -3 & 1 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} + \begin{bmatrix} 1 \\ 2/3 \end{bmatrix}$$

5. Consider the system $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$ given below:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

(a) Find the equilibrium solution, $\mathbf{x}_E$.

SOLUTION: The equilibrium solution for a differential equation is where the derivative is zero.

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 10 \\ 10 \end{bmatrix} \qquad \mathbf{x} = \frac{1}{2-12} \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} -10 \\ -10 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

(b) Show that, if $\mathbf{u} = \mathbf{x} - \mathbf{x}_E$, then the differential equation for $\mathbf{u}$ is: $\mathbf{u}' = A\mathbf{u}$.

SOLUTION: We can show it in general-

$$\mathbf{u}' = \mathbf{x}' - \mathbf{x}_E = A\mathbf{x} + \mathbf{b} - \mathbf{x}_E + A\mathbf{x}_E - A\mathbf{x}_E = A(\mathbf{x} - \mathbf{x}_E) + \mathbf{b} + A(-A^{-1}\mathbf{b}) = A(\mathbf{x} - \mathbf{x}_E)$$

Therefore, $\mathbf{u}' = A\mathbf{u}$.

(c) Solve the differential equation by first solving the DE for $\mathbf{u}$.

TYPO: The eigenvalues/eigenvectors for A are not simple expressions, so write your answer symbolically, assuming two distinct eigenvalues.

SOLUTION:

$$\mathbf{u}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 \quad \Rightarrow \quad \mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 + \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

6. Use the Poincaré Diagram to determine how the origin changes stability by changing α if

$$\mathbf{x}' = \begin{bmatrix} \alpha+1 & \alpha \\ 2 & 1 \end{bmatrix} \mathbf{x}$$

SOLUTION: To use the Poincaré Diagram, we look at expressions for the trace, determinant and discriminant and determine where each is positive/negative/zero. In this situation,

$$\mathrm{Tr}(A) = \alpha + 2 \qquad \det(A) = 1 - \alpha \qquad \Delta = \alpha^2 + 8\alpha$$

Performing a sign chart analysis (at the bottom, divide the α number line by where each quantity is zero)

$\alpha+2$	$-$	$-$	$+$	$+$	$+$
$1-\alpha$	$+$	$+$	$+$	$+$	$-$
$\alpha(\alpha+8)$	$+$	$-$	$-$	$+$	$+$
	$\alpha < -8$	$-8 < \alpha < -2$	$-2 < \alpha < 0$	$0 \le \alpha < 1$	$\alpha > 1$

We can now read off the results, from left to right:

- If $\alpha < -8$, we have a sink.
- If $\alpha = -8$, we have a degenerate sink.
- If $-8 < \alpha < -2$, we have a spiral sink.

- If $\alpha = -2$, we have a center.
- If $-2 < \alpha < 0$, we have a spiral source.
- If $\alpha = 0$, we have a degenerate source.
- If $0 < \alpha < 1$, we have a source.
- If $\alpha = 1$, we have a line of unstable fixed points.
- If $\alpha > 1$, we have a saddle.

7. Let F be given below, and linearize it at the given value.

   (a) $\mathbf{F}(t) = \begin{bmatrix} t^2 + 3t + 2 \\ \sqrt{t+1} + 1 \\ \sin(t) \end{bmatrix} \qquad \text{at} \qquad t = 0$

   SOLUTION: The "derivative" in this case is computed element-wise, so that:

   $$\mathbf{F}(0) = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \qquad \mathbf{F}'(0) = \begin{bmatrix} 3 \\ 1/2 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{L}(t) = \mathbf{F}(0) + \mathbf{F}'(0)t = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1/2 \\ 1 \end{bmatrix}$$

   (b) $f(x, y, z) = x^2 + 3x + 2y + 4z - 2 \qquad \text{at} \qquad (x, y, z) = (1, -1, 1)$

   In this case, the linearization is given by:

   $$L(x, y, z) = f(1, -1, 1) + f_x(1, -1, 1)(x-1) + f_y(1, -1, 1)(y+1) + f_z(1, -1, 1)(z-1)$$

   Substituting everything in, we get:

   $$L(x, y, z) = 4 + 5(x-1) + 2(y+1) + 4(z-1)$$

   (c) $\mathbf{F}(x, y) = \begin{bmatrix} x^2 + 3xy - y + 1 \\ y^2 + 2xy + x^2 - 1 \end{bmatrix} \qquad \text{at } (x, y) = (1, 0)$

   In this case, the linearization is given by the following, if we think of $f(x, y) = x^2 + 3xy - y + 1$ and $g(x, y) = y^2 + 2xy + x^2 - 1$:

   $$\mathbf{L}(x, y) = \mathbf{F}(1, 0) + \begin{bmatrix} f_x(1,0) & f_y(1,0) \\ g_x(1,0) & g_y(1,0) \end{bmatrix} \begin{bmatrix} x-1 \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x-1 \\ y \end{bmatrix}$$

8. For each nonlinear system below, perform a local linear analysis about all equilibria.

   (a) $\begin{aligned} dx/dt &= x - xy \\ dy/dt &= y + 2xy \end{aligned}$

   We should find two equilbria. The origin is a source and the point $(-1/2, 1)$ is a saddle.

(b) $$\begin{aligned} dx/dt &= 1 + 2y \\ dy/dt &= 1 - 3x^2 \end{aligned}$$

We should find two equilibria: $(-\sqrt{3}, -1/2)$ and $(\sqrt{3}, -1/2)$. The first equilibrium is saddle, the second is a center.

Side Remark: In this instance, the full nonlinear system actually has a spiral sink at the second equilibrium, but we would have failed to see it because of the linearization.

9. For each of the systems in question 8, solve them by first computing dy/dx.

   (a) In this case, we have a separable differential equation:

   $$\frac{dy}{dx} = \frac{y(1+2x)}{x(1-y)} \quad \Rightarrow \quad \int \frac{1-y}{y}\,dy = \int \frac{1+2x}{x}\,dx \quad \Rightarrow \quad \ln|y| - y = \ln|x| + 2x + C$$

   (b) In this case, we also have a separable differential equation:

   $$\frac{dy}{dx} = \frac{1-3x^2}{1+2y} \quad \Rightarrow \quad \int 1 + 2y\,dy = \int 1 - 3x^2\,dx \quad \Rightarrow \quad y + \frac{1}{2}y^2 = x - x^3 + C$$

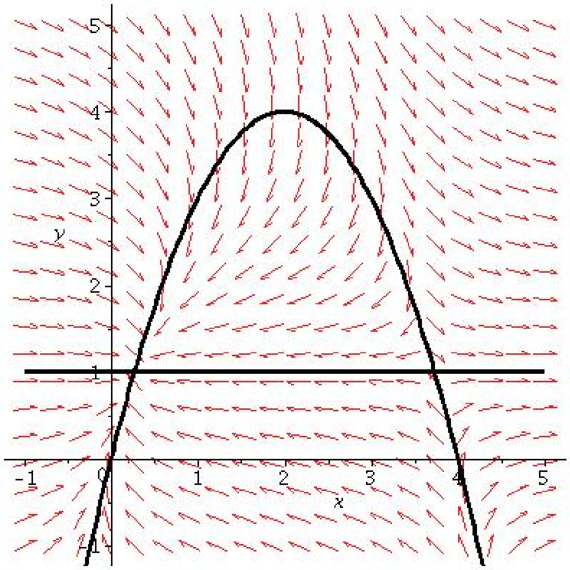
10. For 8(a) above, if x and y were two populations, what kinds of assumptions are being made to result in these differential equations?

    SOLUTION: In the absence of the other, both populations experience exponential growth. In the presence of interactions between them, x suffers and y benefits (perhaps y is eating x!).

11. Given $x' = f(x, y)$ and $y' = g(x, y)$, then a **nullcline** is a curve where f or g is 0. Note that an equilibrium is where the nullclines intersect.

    If $x' = -4x + y + x^2$ and $y' = 1 - y$, then graph the nullclines, taking note of the equilbrium solutions. Is there an area in your drawing where $x' < 0$ and $y' < 0$? Make note of it.

    SOLUTION: See the figure below. The region of interest is above the line and inside the parabola.



12. Is the following system an example of predator-prey or competing species? In either case, perform a local linear analysis:

$$\begin{aligned} x' &= x(1-0.5y) \\ y' &= y(-0.75+0.25x) \end{aligned}$$

SOLUTION: This is an example of predator-prey (x is the prey). There are two equilibria: $(0,0)$ and $(3,2)$. When we linearize about the origin, we get a saddle, and when we linearize about $(3,2)$, we get a center (which in fact does not persist in the full nonlinear case).

13. Assume the temperature of a roast in the oven increases at a rate proportional to the difference between the oven temperature (set to 400) and the roast temperature. If the roast enters the oven at 50 degrees, and is measured one hour later to be 90, when will the roast reach the FDA safe temperature of 160? (Hint: Write down, then solve the difference equation).

TYPO: "difference equation" should be "differential equation"- The question gives you the model for Newton's Law of Cooling.

Let $R(t)$ be the temp of the roast at time t. Then we're told that, if T is the constant temp of the oven, then

$$\frac{dR}{dt} = -k(R-T) \quad \Rightarrow \quad R' = -kR + kT$$

Initially, if we solve the first order linear DE, we get:

$$R(t) = C\mathrm{e}^{-kt} + 400$$

Solving for C, we get $C = -350$. Then we solve the equation: $90 = -350e^{-k} + 400$ for the constant k, and the solution is then:

$$R(t) = -350e^{\ln(31/35)\,t} + 400$$

Solving for t when the roast is 160, $t \approx 3.1$ hours- so the roast will be ready in approximately 3 hours, 6 minutes.

14. Consider the sytem of differential equations below.

$$\frac{dx}{dt} = x(1 - x - y), \qquad \frac{dy}{dt} = y\left(\alpha - y - \frac{1}{2}x\right)$$

where you may assume that $x \geq 0, y \geq 0$, and $\alpha \geq 0$.

(a) Draw the nullclines, and locate the equilbria graphically. You might note that $\alpha = 1$ and $\alpha = 1/2$ are some special cases to consider.

SOLUTION: For the equilibria, we'll solve the first equation first:

$$x(1 - x - y) = 0 \quad \Rightarrow \quad x = 0 \quad \text{or} \quad y = 1 - x$$

Now, if $x = 0$ in Equation 1, then we go to Equation 2:

$$y(\alpha - y - 0) = 0 \quad \Rightarrow \quad y = 0 \text{ or } y = \alpha$$

So far, we have $(0, 0)$ and $(0, \alpha)$ as two equilibria (valid if $\alpha > 0$).

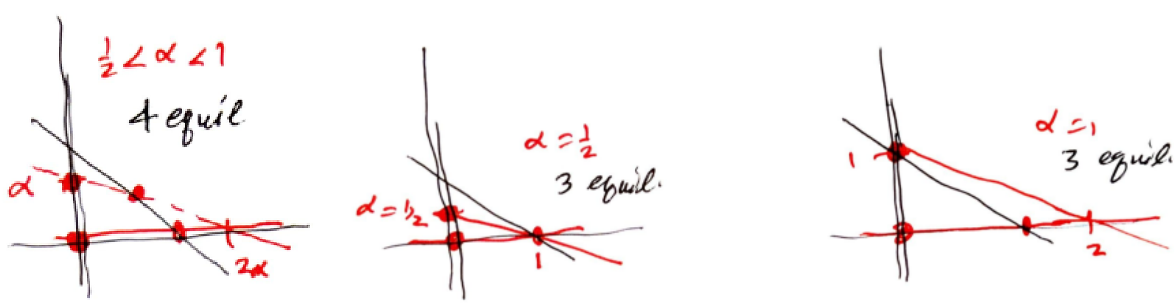
Next, we go back to Equation 1 so that $y = 1 - x$, and see what happens in Equation 2:

$$(1 - x)\left(\alpha - (1 - x) - \frac{1}{2}x\right) = 0 \quad \Rightarrow \quad x = 1 \quad \text{or} \quad x = 2(1 - \alpha)$$

Using the fact that $y = 1 - x$, this gives us more equilibria- Listing them all so far:

$$(0, 0), \quad (0, \alpha), \quad (1, 0), \quad (2(1 - \alpha), 2\alpha - 1)$$

We note that the last equilibrium will be valid only if $\frac{1}{2} \leq \alpha \leq 1$ (otherwise, either x or y is negative). With that in mind, here are some quickly sketched cases to consider.



(b) Linearize the system.

Typo: The second question is a little vague. It should say "Find the Jacobian matrix for the system", which is:

$$\begin{bmatrix} 1-y-2x & -x \\ -\frac{1}{2}y & \alpha - 2y - \frac{1}{2}x \end{bmatrix}$$

(c) Analyze what happens at the origin in terms of α using the Poincare Diagram.

SOLUTION: At the origin, we simply get the following matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}$$

so that the trace is $1+\alpha$ and the determinant is α. Further, the discriminant is can be factored to $(\alpha-1)^2$, so that if $\alpha > 0$, the origin is always a SOURCE (at $\alpha = 1$, it is a degenerate source).

(d) If $\alpha = \frac{3}{4}$, linearize at $(1/2, 1/2)$ and give the results.

At $(1/2, 1/2)$ where $\alpha = 3/4$, we should find that the trace is -1, the determinant is $1/8$ and the discriminant is $1/2$, so the point $(1/2, 1/2)$ is a SINK. By the way, here is the Jacobian matrix:

$$\begin{bmatrix} -1/2 & -1/2 \\ -1/4 & -1/2 \end{bmatrix}$$

15. Consider the IVP:

$$\begin{aligned} \frac{dx}{dt} &= -x + 3z \\ \frac{dy}{dt} &= -y + 2z \\ \frac{dz}{dt} &= x^2 - 2z \end{aligned}$$

where $x(0) = 0, y(0) = 1/2$, and $z(0) = 3$. Further, we want the solution for $0 \le t \le 1.5$

(a) Numerically solve the system of equations given above, and print/save the file you used for the derivatives. First use Euler's method (forward), with step size 0.1, and plot the result.

For a sample solution, see the figures on the next page.

(b) Repeat the previous solution, but use our own Runge-Kutta algorithm from class with step size 0.1. Plot the result.

The solution to this is identical to the previous one, except in using our Runge-Kutta algorithm. To provide your solution, you would provide something like what we did for the previous problem.

(c) Repeat the solution, but use Octave's built-in function `ode23`, and plot the result.

To show your solution, show something like the solution to the previous problem- A screen shot showing your work would suffice.

(d) Linearize the system at the origin, then use Octave to find the eigenvalues. Considering the output, is the origin a sink, a source, or something else?

In this case, I just wanted to remind you that Octave can compute these. In the diagonal matrix below, the eigenvalues are along the diagonal (and the eigenvectors are, in order, in matrix V).

Because all eigenvalues are negative, the origin is a SINK.

```
octave:1> A=[-1 0 3;0 -1 2;0 0 -2];
octave:2> [V,D]=eig(A)
V =

   1.0000        0  -0.8018
        0   1.0000  -0.5345
        0        0   0.2673

D =

Diagonal Matrix

  -1   0   0
   0  -1   0
   0   0  -2
```

Output for Exercise 15. First, here are the derivatives:

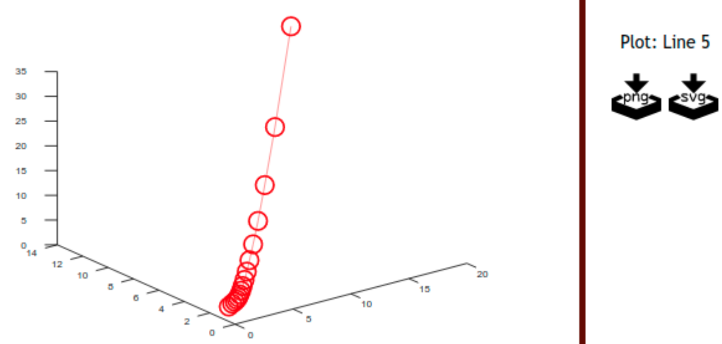
```
myfunc3.m

function dy=myfunc3(t,y)

dy=zeros(size(y));
dy(1)=-y(1)+3*y(3);
dy(2)=-y(2)+2*y(3);
dy(3)=y(1)^2-2*y(3);

end
```

Next is the output from the software, with a plot:




```
Vars
# ans
# h
[1x16] t
# t0
# tf
[16x3] y
[3x1] yin

octave:1> yinit=[0;1/2;3]; t0=0; tf=1.5; h=0.1;
octave:2> [t,y]=euler_forward(@myfunc3,t0,yinit,tf,h);
warning: function name 'myfunc' does not agree with function
filename '/home/oo/myfunc3.m'
octave:3> [t,y]=euler_forward(@myfunc3,t0,yinit,tf,h);
octave:4> whos
Variables visible from the current scope:

variables in scope: top scope

   Attr Name        Size                     Bytes  Class
   ==== ====        ====                     =====  =====
        ans         1x1                          8  double
        h           1x1                          8  double
        t           1x16                        24  double
        t0          1x1                          8  double
        tf          1x1                          8  double
        y          16x3                        384  double
        yinit       3x1                         24  double

Total is 71 elements using 464 bytes

octave:5> plot3(y(:,1),y(:,2),y(:,3),'ro-')
```

1