HW Solutions from Feb 26 (Linear Algebra notes)

1. Let the subspace H be formed by the span of the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ given below. Given the point $\boldsymbol{x}_1, \boldsymbol{x}_2$ below, find which one belongs to H, and if it does, give its coordinates. (NOTE: The basis vectors are NOT orthonormal)

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad \boldsymbol{x}_1 = \begin{bmatrix} 7 \\ 4 \\ 0 \end{bmatrix} \quad \boldsymbol{x}_2 = \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}$$

SOLUTION: Rather than row-reduce twice, we'll do it once on the augmented matrix below.

$$\left[\begin{array}{rr|rr} 1 & 2 & 7 & 4 \\ 2 & -1 & 4 & 3 \\ -1 & 1 & 0 & -1 \end{array}\right] \to \left[\begin{array}{rr|rr} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

How should this be interpreted? The second vector, $\boldsymbol{x}_2$ is in H, as it can be expressed as $2\boldsymbol{v}_1 + \boldsymbol{v}_2$. Its low dimensional representation is thus $[2, 1]^T$.

The first vector, $\boldsymbol{x}_1$, cannot be expressed as a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, so it does not belong to H.

3. Let the subspace G be the plane defined below, and consider the vector $\boldsymbol{x}$, where:

$$G = \left\{ \alpha_1 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \text{ such that } \alpha_1, \alpha_2 \in \mathbb{R} \right\} \qquad \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

(a) Find the matrix (UU^T in our notes) that takes an arbitrary vector and projects it (orthogonally) to the plane G.

SOLUTION: The matrix U consists of the two column vectors above, but you MUST normalize them:

$$U = \begin{bmatrix} 1/\sqrt{14} & 3/\sqrt{10} \\ 3/\sqrt{14} & -1/\sqrt{10} \\ -2/\sqrt{14} & 0 \end{bmatrix} \qquad UU^T = \frac{1}{35} \begin{bmatrix} 34 & -3 & -5 \\ -3 & 26 & -15 \\ -5 & -15 & -10 \end{bmatrix}$$

(b) Find the orthogonal projection of the given $\boldsymbol{x}$ onto the plane G.

$$UU^T\boldsymbol{x} = \frac{1}{35} \begin{bmatrix} 24 \\ -33 \\ 15 \end{bmatrix}$$

(c) Find the distance from the plane G to the vector $\boldsymbol{x}$.

$$\|\boldsymbol{x} - UU^T\boldsymbol{x}\| = \sqrt{\frac{11}{35}^2 + \frac{33}{35}^2 + \frac{11}{7}^2} \approx 3.511$$

6. Let $\boldsymbol{a} = [-1, 3]^T$. Find a square matrix P so that $P\boldsymbol{x}$ is the orthogonal projection of $\boldsymbol{x}$ onto the span of $\boldsymbol{a}$.

   SOLUTION: Let matrix U be just the one column with (normalized!) vector $\boldsymbol{a}$. Then:

$$U = \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \quad \Rightarrow \quad P = UU^T = \frac{1}{10} \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$$

7. Refer to one of the programming languages (Matlab/Python/R), and reproduce finding an arbitrary 10×4 matrix with orthonormal columns. Use a random $\boldsymbol{x} \in \mathbb{R}^{10}$, and first find the coordinates of $\boldsymbol{x}$ with respect to the four columns in Q, then compute the orthogonal projection of $\boldsymbol{x}$ into the subspace spanned by the first four columns of Q.

   SOLUTION: Given the computed matrix Q and vector $\boldsymbol{x}$, you should have (in your programming language):

   - The coordinates of $\boldsymbol{x}$: $Q^T\boldsymbol{x}$
   - The orthogonal projection: $QQ^T\boldsymbol{x}$.

8. To prove that we have an *orthogonal* projection, the vector $\text{Proj}_u(\boldsymbol{x}) - \boldsymbol{x}$ should be orthogonal to $\boldsymbol{u}$. Use this definition to show that our earlier formula was correct- that is,

$$\text{Proj}_u(\boldsymbol{x}) = \frac{\boldsymbol{x} \cdot \boldsymbol{u}}{\boldsymbol{u} \cdot \boldsymbol{u}} \boldsymbol{u}$$

   is the orthogonal projection of $\mathbf{x}$ onto $\mathbf{u}$.

   SOLUTION: We show that $(\text{Proj}_u(\boldsymbol{x}) - \boldsymbol{x}) \cdot \boldsymbol{u} = 0$

$$\left(\frac{\boldsymbol{x} \cdot \boldsymbol{u}}{\boldsymbol{u} \cdot \boldsymbol{u}} \boldsymbol{u} - \boldsymbol{x} \right) \cdot \boldsymbol{u} = \frac{\boldsymbol{x} \cdot \boldsymbol{u}}{\boldsymbol{u} \cdot \boldsymbol{u}} (\boldsymbol{u} \cdot \boldsymbol{u}) - \boldsymbol{x} \cdot \boldsymbol{u} = \boldsymbol{x} \cdot \boldsymbol{u} - \boldsymbol{x} \cdot \boldsymbol{u} = 0$$