HW Solutions from Mar 1

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1. Show that Null(A) $\perp$ Row(A). You must show that, for arbitrary $\boldsymbol{x}_1 \in$ Null(A) and $\boldsymbol{x}_2 \in$ Row(A), we have the inner product is $(\boldsymbol{x}_1, \boldsymbol{x}_2) = 0$. Hint: Write A in terms of its rows.

   If we write A in terms of rows, $A = [r_1, r_2, \ldots, r_m]^T$, then the i^{th} component of $A\boldsymbol{x}$ can be written as $\boldsymbol{r}_i\boldsymbol{x} = 0$. Thus, if $\boldsymbol{x}$ is in the nullspace of A, then it is orthogonal to every row of A. If $\boldsymbol{x}_2$ is a linear combination of the rows of A, then (by the properties of the dot product), it will be orthogonal to a vector from the nullspace of A.

   This argument also goes in reverse.

3. Show that multiplication by an orthogonal matrix preserves lengths: $\|\mathbb{Q}\boldsymbol{x}\|_2 = \|\boldsymbol{x}\|_2$ (Hint: Use properties of inner products). Conclude that multiplication by $\mathbb{Q}$ represents a rigid rotation.

$$\|\mathbb{Q}\boldsymbol{x}\|_2 = \boldsymbol{x}^T Q^T Q\boldsymbol{x} = \|\boldsymbol{x}\|_2$$

4. Prove the Pythagorean Theorem by induction: Given a set of n orthogonal vectors $\{\boldsymbol{x}_i\}$

$$\|\sum_{i=1}^{n} \boldsymbol{x}_i\|_2^2 = \sum_{i=1}^{n} \|\boldsymbol{x}_i\|_2^2$$

   The case with $n = 1$ is trivial. The case with $n = 2$ is straightforward. Assume true for $k = n$,

$$\|\sum_{i=1}^{n} \boldsymbol{x}_i\|_2^2 = \sum_{i=1}^{n} \|\boldsymbol{x}_i\|_2^2$$

   Show true for $k = n + 1$:

$$\|\sum_{i=1}^{n+1} \boldsymbol{x}_i\|_2^2 = \left(\sum_{i=1}^{n} \boldsymbol{x}_i + \boldsymbol{x}_{n+1}\right)^T \left(\sum_{i=1}^{n} \boldsymbol{x}_i + \boldsymbol{x}_{n+1}\right) =$$

$$\left(\sum_{i=1}^{n} \boldsymbol{x}_i\right)^T \left(\sum_{i=1}^{n} \boldsymbol{x}_i\right) + \left(\sum_{i=1}^{n} \boldsymbol{x}_i\right) \boldsymbol{x}_{n+1}^T + \boldsymbol{x}_{n+1} \left(\sum_{i=1}^{n} \boldsymbol{x}_i\right)^T + \boldsymbol{x}_{n+1}^T \boldsymbol{x}_{n+1}$$

6. The Orthogonal Decomposition Theorem: if $\boldsymbol{x} \in \mathbb{R}^n$ and W is a (non-zero) subspace of $\mathbb{R}^n$, then $\boldsymbol{x}$ can be written *uniquely* as

$$\boldsymbol{x} = \boldsymbol{w} + \boldsymbol{z}$$

   where $\boldsymbol{w} \in W$ and $\boldsymbol{z} \in W^{\perp}$.

   To prove this, let $\{\boldsymbol{u}_i\}_{i=1}^{p}$ be an orthonormal basis for W, define $\boldsymbol{w} = (\boldsymbol{x}, \boldsymbol{u}_1)\boldsymbol{u}_1 + \ldots + (\boldsymbol{x}, \boldsymbol{u}_p)\boldsymbol{u}_p$, and define $\boldsymbol{z} = \boldsymbol{x} - \boldsymbol{w}$. Then:

(a) Show that $\boldsymbol{z} \in W^\perp$ by showing that it is orthogonal to every $\boldsymbol{u}_i$.

Let $z = x - \alpha_1 u_1 - \alpha_2 u_2 - \ldots - \alpha_p u_p$, where $\alpha_j = x \cdot u_j$. Then we write out $z \cdot u_k$ for an arbitrary $k = 1, 2, \ldots, p$:

$$x \cdot u_k - \alpha_1 u_1 \cdot u_k - \ldots - \alpha_p u_p \cdot u_k = x \cdot u_k - \alpha_k = 0$$

(b) To show that the decomposition is unique, suppose it is not. That is, there are two decompositions:

$$\boldsymbol{x} = \boldsymbol{w}_1 + \boldsymbol{z}_1, \quad \boldsymbol{x} = \boldsymbol{w}_2 + \boldsymbol{z}_2$$

Show this implies that $\boldsymbol{w}_1 - \boldsymbol{w}_2 = \boldsymbol{z}_2 - \boldsymbol{z}_1$, and that this vector is in both W and $W^\perp$. What can we conclude from this?

The only vector that is in both W and $W^\perp$ is the zero vector. Therefore, $\boldsymbol{w}_1 = \boldsymbol{w}_2$, and $\boldsymbol{z}_2 = \boldsymbol{z}_1$.