HW Solutions Assigned Mar 3

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1. Show that if λ is an eigenvalue of A^TA, then λ is also an eigenvalue of AA^T.

   Hint: If λ is an eigenvalue of A^TA with eigenvector $\mathbf{v}$, then $A^TA\mathbf{v} = \lambda\mathbf{v}$. Now what equation must be true if λ is an eigenvalue of AA^T?

   SOLUTION: If λ is an eigenvalue of A^TA with eigenvector $\mathbf{v}$, then

$$A^TA\mathbf{v} = \lambda\mathbf{v} \quad \Rightarrow \quad AA^TA\mathbf{v} = \lambda A\mathbf{v} \quad \Rightarrow \quad AA^T\boldsymbol{u} = \lambda\boldsymbol{u}$$

2. There is a beautiful relationship between the column vectors of U and V. We will assume that A is $m \times n$ with rank k, and $A = U\Sigma V^T$ is the SVD of A. Then:

$$A\boldsymbol{v}_i = \sigma_i\boldsymbol{u}_i \qquad \text{and} \qquad A^T\boldsymbol{u}_i = \sigma_i\boldsymbol{v}_i$$

   NOTE: The importance of this is that, if you know the left singular vectors, then you can compute the right singular vectors.

   HINT: You might start with $A\boldsymbol{v}_i$, then recall that $A = U\Sigma V^T$, which can be written as a sum of k matrices. The proof of the second equation is similar.

8. Compute the SVD by hand of the following matrices:

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

   (Hint: A^TA and AA^T are symmetric matrices.)

   SOLUTION: I'm giving the solution below- If you need help with the steps in the computation, please ask!

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^T$$

   and

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$