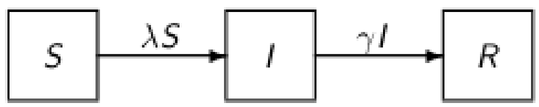
The S-I-R Model for Spread of Disease[1]

We divide the population into three groups:

- Susceptible individuals ("susceptibles"), $S(t)$
- Infected individuals, $I(t)$.
- Recovered individuals, $R(t)$.

We first assume that only susceptible indivduals become infected, and only individuals that have been infected become recovered. That is, we assume a certain "flow" of the population:



Other Model Assumptions

- Population size is large and constant, N. Therefore, for every time t,

$$S(t) + I(t) + R(t) = N$$

- The last thing means that we are not taking into account any births, deaths, immigration or emigration.
- No latent period (immediately go from one class to the other).
- Homogeneous mixing.
- Infection rate is proportional to the fraction of infected:

$$\lambda = \beta \frac{I}{N}$$

- Recovery rate is constant, γ.

[1]Notes from Jan Medlock, Clemson

The ODEs

$$\frac{dS}{dt} = -\lambda S(t)$$
$$\frac{dI}{dt} = \lambda S(t) - \gamma I(t)$$
$$\frac{dR}{dt} = \gamma I(t)$$

Using the 5th assumption, we can rewrite these equations as:

$$\frac{dS}{dt} = -\beta \frac{I}{N} S(t) \tag{1}$$
$$\frac{dI}{dt} = \beta \frac{I}{N} S(t) - \gamma I(t) \tag{2}$$
$$\frac{dR}{dt} = \gamma I(t) \tag{3}$$

We note that assumption 1 (Population is large and constant and equal to N) really makes the equation for R irrelevant, so that we could simply write:

$$\frac{dS}{dt} = -\beta \frac{I}{N} S(t)$$
$$\frac{dI}{dt} = \beta \frac{I}{N} S(t) - \gamma I(t)$$
$$R(t) = N - S(t) - I(t)$$

A Discussion of Parameters

The value of γ represents the average period of infectiousness. For example, if the period is three days, that would suggest that $\gamma = 1/3$. Similarly, β is a rate of infection in the sense that, if an infected person generally makes an infecting contact every two days, then that would suggest $\beta = 1/2$.

An important quantity: $\mathcal{R}_0$

We may note that at the beginning of the epidemic, the susceptible population is the entire population ($S(t) \approx N$), so that the rate of change of the infected population is then approximately

$$\frac{dI}{dt} \approx (\beta - \gamma) I(t)$$

Therefore, if $\beta - \gamma > 0$, $I(t)$ increases.

You may have noted a tremendous amount of news coverage of the model parameter $\mathcal{R}_0$, which is the **basic reproductive number**, which is defined as:

$$\mathcal{R}_0 = \frac{\beta}{\gamma}$$

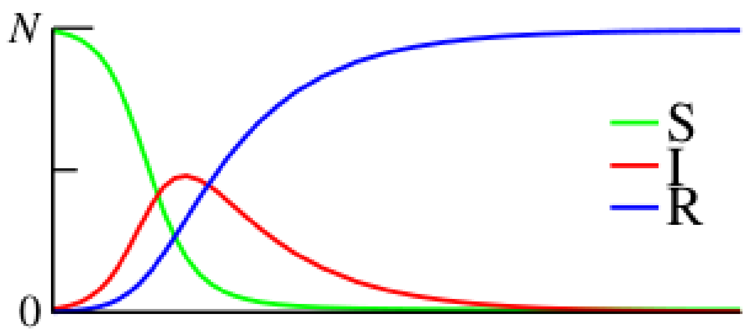
With our previous observation, if $\mathcal{R}_0 > 1$, then $I(t)$ increases, and we have an epidemic.

Here are some values of $\mathcal{R}_0$ for several diseases:

Disease	$\mathcal{R}_0$
Measles	12-18
Chickenpox	10-12
Polio	5-7
COVID-19	3.3-5.7
Common Cold	2-3
Ebola	1.5-1.9
Seasonal Flu	0.9-2.1

Numerical Solvers for SIR

If we use a numerical solver for the SIR model, we'll get solution curves that generally look like the following:



In the current pandemic, a popular saying early on was that we needed to lower the curve- In this case, it would be to take out the local maximum, and instead have something that goes straight towards zero. (We'll experiment with this a bit in the exercises).

Linearization and SIR

If we look at only the first two equations (since $R(t)$ is determined from those), then

$$\frac{dS}{dt} = -\beta \frac{I}{N} S = 0 \tag{4}$$

$$\frac{dI}{dt} = \beta \frac{I}{N} S - \gamma I = 0 \tag{5}$$

This gives two equilibria: $(S, I) = (N, 0)$ and $(S, I) = (0, 0)$. The first is **disease-free equilibrium**, and the second is after the pandemic when $R = N$.

We should find that, after linearizing about $(N, 0)$, the Jacobian matrix is given by

$$\begin{bmatrix} 0 & -\beta \\ 0 & \beta - \gamma \end{bmatrix} \quad \Rightarrow \quad \lambda = 0, \beta - \gamma$$

Homework Questions

1. Show that, if we define:

$$\begin{aligned} s(t) &= S(t)/N, && \text{the susceptible fraction of the population,} \\ i(t) &= I(t)/N, && \text{the infected fraction of the population, and} \\ r(t) &= R(t)/N, && \text{the recovered fraction of the population.} \end{aligned}$$

   then we can change the differential equations: (Matlab code given also)

$$\begin{aligned} \frac{ds}{dt} &= -\beta\, s\, i \\ \frac{di}{dr} &= \beta\, s\, i\, - \gamma\, i \\ \frac{dr}{dt} &= \gamma\, i \end{aligned}$$

```
function dy=SIREqns(t,y)
  beta=1/2; gamma=1/3;
  dy=zeros(size(y));
  dy(1)=-beta*y(1)*y(2);
  dy(2)=beta*y(1)*y(2)-gamma*y(2);
  dy(3)=gamma*y(2);
```

2. **Experimenting with parameters**: In this question, set up the DE solver to solve the SIR model using the percentages in the previous question, and create a file, `SIREqns.m` that contain the differential equations. Then you can use `ode23` to experiment with the numerical solutions.

   (a) Keep $\gamma = 1/3$, and plot the graph of $i(t)$ with several different values of β ranging between 0.5 and 2.0. Comment on what you see.

   (b) Keep $\beta = 1/2$, and plot the graph of $i(t)$ with several different values of γ between 0.1 and 0.6. Comment on what you see.

   (c) There is a change in the character of the graph of $i(t)$ near one end of the suggested range for γ in the previous experiment. What is the change, and where does it occur?

   (d) Could we have predicted the change discussed in the previous problem?

3. Show that the linearized system gives the Jacobian matrix shown.

4. Solve the linearized system about the point $(N, 0)$:

$$\begin{aligned} \frac{dS}{dt} &= -\beta I \\ \frac{dI}{dt} &= (\beta - \gamma) I \end{aligned}$$

   (Hint: Solve for I first).