# Solutions to Review questions, Chapter 6

1. Find the Laplace transform of the solution the heat equation:

$$\begin{aligned}
&u_t = u_{xx}, \quad x > 0, t > 0\\
&u(x,0) = 0\\
&u(0,t) = 1\\
&\lim_{x\to\infty} u(x,t) = 0, \quad t > 0
\end{aligned}$$

   SOLUTION: Apply the transform to both sides of the equation (we're transforming $t$ to $s$):

$$\mathcal{L}(u_t(x,t)) = \mathcal{L}(u_{xx}(x,t)) \quad \Rightarrow \quad s\overline{U}(x,s) - u(x,0) = \overline{U}_{xx}(x,s) \quad \Rightarrow \quad \overline{U}_{xx}(x,s) - s\overline{U}(x,s) = 0$$

   This is a standard second order DE (think: $y''(x) - sy(x) = 0$), so the characteristic equation is given by $r^2 - s = 0$, or $r = \pm\sqrt{s}$, and the solution is:

$$\overline{U}(x,s) = C_1(s)\mathrm{e}^{\sqrt{s}x} + C_2(s)\mathrm{e}^{-\sqrt{s}\,x}$$

   Since the limit of $u(x,t)$ is 0, so is the limit of $U(x,s)$ (in $x$), so we set $C_1(s) = 0$. Further,

$$\hat{U}(0,s) = \mathcal{L}(u(0,t)) = \mathcal{L}(1) = \frac{1}{s}$$

   from which we determine that $C_2(s) = \frac{1}{s}$, and the solution is:

$$\overline{U}(x,s) = \frac{1}{s}\mathrm{e}^{-\sqrt{s}\,x}$$

2. Use the Laplace transform to solve the problem:

$$\begin{aligned}
&u_t + 2u_x = 0, \quad x > 0, t > 0\\
&u(x,0) = 3\\
&u(0,t) = 5
\end{aligned}$$

   SOLUTION: (This was problem 1(a) in 6.1)

$$\mathcal{L}(u_t(x,t)) + 2\mathcal{L}(u_x(x,t)) = 0 \quad \Rightarrow \quad s\overline{U}(x,s) - u(x,0) + 2\overline{U}_x(x,s) = 0$$

   Substitute and rewrite as:

$$2\overline{U}_x(x,s) + s\overline{U}(x,s) = 3$$

   Side computation: Think of this as the ODE ($y$ as a function of $x$):

$$2y'' + sy = 3 \quad \Rightarrow \quad y'' + \frac{s}{2}y = \frac{3}{2}$$

The integrating factor is $e^{\frac{s}{2}x}$, so that the ODE above becomes:

$$\left(e^{\frac{s}{2}x}y(x)\right)' = \frac{3}{2}e^{\frac{s}{2}x} \quad \Rightarrow \quad e^{\frac{s}{2}x}y(x) = \frac{3}{2}\frac{2}{s}e^{\frac{s}{2}x} + C(s) = \frac{3}{s}e^{\frac{s}{2}x} + C(s)$$

Therefore,

$$y(x) = \frac{3}{s} + C(s)e^{-\frac{x}{2}s}$$

Putting this back as $\overline{U}(x,s)$, and using Table #7 to do the inverse transform, we get:

$$u(x,t) = 3 + 2h(t - x/2)$$

where $h(t)$ is the Heaviside function.

3. Use the Laplace transform to solve the wave equation for the transformed solution.

$$\begin{array}{l} u_{tt} = 9u_{xx}, x > 0, t > 0 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \\ u(0,t) = f(t) \\ lim_{x\to\infty} u(x,t) = 0 \end{array}$$

SOLUTION: Take the Laplace transform of both sides first:

$$s^2\overline{U}(x,s) - su(x,0) - u_t(x,0) = 9\overline{U}_{xx}(x,s) \quad \Rightarrow \quad 9\overline{U}_{xx} - s^2\overline{U} = 0$$

The characteristic equation is $9r^2 - s^2 = 0$, so

$$r = \pm\frac{s}{3} \quad \Rightarrow \quad \overline{U}(x,s) = C_1(s)e^{\frac{sx}{3}} + C_2(s)e^{-\frac{sx}{3}}$$

With the limit being zero, that means $C_1(s) = 0$. Using the last condition, we see that $C_2(s) = F(s)$, and

$$\overline{U}(x,s) = F(s)e^{-\frac{sx}{3}}$$

4. Use the Laplace transform to find the transform of the solution to:

$$\begin{array}{l} u_t = u_{xx}, \quad x > 0, t > 0 \\ u(x,0) = 0 \\ u_x(0,t) = 1 \\ \lim_{x\to\infty} u(x,t) = 0, \quad t > 0 \end{array}$$

SOLUTION: Generally the same as the previous two problems:

$$s\overline{U}(x,s) - u(x,0) = \overline{U}_{xx}(x,s)$$

We get that the characteristic equation is $r^2 - s = 0$, so $r = \pm\sqrt{s}$, and the general solution is

$$\overline{U}(x,s) = C_1(s)\mathrm{e}^{\sqrt{s}\,x} + C_2(s)\mathrm{e}^{-\sqrt{s}\,x}$$

The limit once again is zero as $x \to \infty$, so $C_1(s) = 0$. That leaves us with

$$\overline{U}(x,s) = C_2(s)\mathrm{e}^{-\sqrt{s}\,x}$$

To get the last function, differentiate $\overline{U}$ with respect to $x$:

$$\overline{U}_x(x,s) = -\sqrt{s}C_2(s)\mathrm{e}^{-\sqrt{s}\,x}$$

and using the last piece of data, $\overline{U}_x(0,s) = \frac{1}{s}$. Therefore,

$$\overline{U}(x,s) = -\frac{1}{s^{3/2}}\mathrm{e}^{-\sqrt{s}\,x}$$

We could invert this using Table Entry 11 and 12, but the question asked us only to find the transform of the solution.

5. Compute the Fourier sine and cosine transform of $\mathrm{e}^{-cx}$. Hint: You can do them both at once.

   SOLUTION: We write both the cosine and the sine transforms using the complex exponential:

$$F_c(\alpha) + iF_s(\alpha) = \frac{2}{\pi}\int_0^\infty f(x)(\cos(\alpha x) + i\sin(\alpha x))\,dx$$

This integral can be put in complex form (and put in $f(x) = \mathrm{e}^{-cx}$:

$$= \frac{2}{\pi}\int_0^\infty \mathrm{e}^{-cx}\mathrm{e}^{i\alpha x}\,dx = \frac{2}{\pi}\int_0^\infty \mathrm{e}^{-(c-i\alpha)x}\,dx = \frac{2}{\pi}\left(\frac{-1}{c-i\alpha}\mathrm{e}^{-(c-i\alpha)x}\right|_0^\infty$$

What happens to the exponential as $x \to \infty$? We can consider the modulus of the solution:

$$|\mathrm{e}^{-cx}\mathrm{e}^{i\alpha x}| = |\mathrm{e}^{-cx}||\cos(\alpha x) + i\sin(\alpha x)| = |\mathrm{e}^{-cx}|$$

We see that $|\mathrm{e}^{i\alpha x}| = 1$, and convergence depends on the real part of the exponential, $\mathrm{e}^{-cx}$, which converges (to zero) when $c > 0$. With that in mind, we can now take the limit:

$$\frac{2}{\pi}\left(\frac{-1}{c-i\alpha}\mathrm{e}^{-(c-i\alpha)x}\right|_0^\infty = \frac{2}{\pi}\left(\frac{-1}{c} - i\alpha\right)(0-1) = \frac{2}{\pi}\frac{1}{c-i\alpha}$$

Now, we want to rationalize the denominator:

$$\frac{1}{c-i\alpha}\frac{c+i\alpha}{c+i\alpha} = \frac{c}{c^2+\alpha^2} + i\frac{\alpha}{c^2+\alpha^2}$$

From which we get:

$$F_c(\alpha) = \frac{2}{\pi}\frac{c^2}{c^2+\alpha^2} \qquad F_s(\alpha) = \frac{2}{\pi}\frac{\alpha}{c^2+\alpha^2}$$

6. Find an expression for the Fourier sine transform of $f'(x)$.

SOLUTION: Define the Fourier sine/cosine transform of $f(x)$ as $F_s(\alpha)$ and $F_c(\alpha)$, respectively. Then:

$$\frac{2}{\pi}\int_0^\infty f'(x)\sin(\alpha x)\,dx$$

$$\begin{array}{ccc} + & \sin(\alpha x) & f'(x) \\ - & \alpha\cos(\alpha x) & f(x) \end{array} \quad \Rightarrow \quad \left(\frac{2}{\pi}f(x)\sin(\alpha x)\right|_0^\infty - \alpha\frac{2}{\pi}\int_0^\infty f(x)\cos(\alpha x)\,dx$$

For the limit, we'll assume $f(x) \to 0$ as $x \to \infty$ (because we're assuming $F_s(\alpha)$ and $F_c(\alpha)$ both exist). From this we get that the Fourier sine transform of $f'(x)$ is:

$$\frac{2}{\pi}(0-0) - \alpha F_c(\alpha) = -\alpha F_c(\alpha)$$

This question didn't ask for it, but as a **bonus solution**, here's the Fourier sine transform for $f''(x)$:

$$\frac{2}{\pi}\int_0^\infty f''(x)\sin(\alpha x)\,dx \quad \Rightarrow \quad \begin{array}{ccc} + & \sin(\alpha x) & f''(x) \\ - & \alpha\cos(\alpha x) & f'(x) \\ + & -\alpha^2\sin(\alpha x) & f(x) \end{array}$$

From which we get:

$$\frac{2}{\pi}\left[f'(x)\sin(\alpha x) - \alpha f(x)\cos(\alpha x)\right|_0^\infty - \alpha^2\frac{2}{\pi}\int_0^\infty f(x)\sin(\alpha x)\,dx$$

For the limit, we'll assume that both $f(x)$ and $f'(x)$ go to zero as $x \to \infty$. From this we get that the Fourier sine transform of $f''(x)$ is:

$$\frac{2}{\pi}[(0-0) - \alpha(0 - f(0))] - \alpha^2 F_s(\alpha) = -\alpha^2 F_s(\alpha) + \alpha\frac{2}{\pi}f(0)$$

7. Find an expression for the Fourier cosine transform of $f''(x)$.

SOLUTION: This is similar to the previous problem.

$$\frac{2}{\pi}\int_0^\infty f''(x)\cos(\alpha x)\,dx \quad \Rightarrow \quad \begin{array}{ccc} + & \cos(\alpha x) & f''(x) \\ - & -\alpha\sin(\alpha x) & f'(x) \\ + & -\alpha^2\cos(\alpha x) & f(x) \end{array}$$

From which we get:

$$\frac{2}{\pi}\left[f'(x)\cos(\alpha x) + \alpha f(x)\sin(\alpha x)\right|_0^\infty - \alpha^2\frac{2}{\pi}\int_0^\infty f(x)\cos(\alpha x)\,dx$$

For the limit, we'll assume that both $f(x)$ and $f'(x)$ go to zero as $x \to \infty$. From this we get that the Fourier cosine transform of $f''(x)$ is:

$$\frac{2}{\pi}[(0 - f'(0)) - \alpha(0-0)] - \alpha^2 F_s(\alpha) = -\alpha^2 F_c(\alpha) - \frac{2}{\pi}f'(0)$$

8. Find the transform of the solution (you need to choose sine or cosine) to:

$$\begin{array}{ll} y'' - y = \mathrm{e}^{-2x}, & x \geq 0 \\ y(0) = 1 & \\ \lim_{x\to\infty} y(x) = 0 & \end{array}$$

SOLUTION: This is Example 3, 6.2. Note that we're only asking for the transform of the solution, which is

$$Y(\alpha) = -\frac{2}{\pi}\frac{\alpha}{(\alpha^2+1)(\alpha^2+4)} + \frac{2}{\pi}\frac{\alpha}{\alpha^2+1}$$

9. Find the transform of the solution (you need to choose sine or cosine) to:

$$\begin{array}{ll} u_t = u_{xx}, & x > 0, t > 0 \\ u(x,0) = f(x) & \\ u(0,t) = 0 & \\ \lim_{x\to\infty} u(x,t) = 0 & \end{array}$$

SOLUTION: This is Example 4, 6.2, where we only want the transform of the solution.

$$\overline{U}(\alpha, t) = F_s(\alpha)\mathrm{e}^{-\alpha^2 t}$$

10. Find the transform of the solution (you need to choose sine or cosine) to:

$$\begin{array}{ll} y'' - y = 3\mathrm{e}^{-4x}, & x \geq 0 \\ y'(0) = 0 & \\ \lim_{x\to\infty} y(x) = 0 & \end{array}$$

SOLUTION: This is Problem 4(a) in 6.2, and very similar to problem 8 above.

$$-\alpha^2 Y_c(\alpha) - \frac{2}{\pi}\cdot 0 - Y_c(\alpha) = 3\cdot\frac{2}{\pi}\frac{4}{\alpha^2+16}$$

Therefore,

$$Y_c(\alpha) = -\frac{24}{\pi}\frac{1}{(\alpha^2+1)(\alpha^2+16)}$$

11. Find the transform of the solution (you need to choose sine or cosine) to:

$$\begin{array}{ll} u_t = u_{xx}, & x > 0, t > 0 \\ u(x,0) = f(x) & \\ u_x(0,t) = 0 & \\ \lim_{x\to\infty} u(x,t) = 0 & \end{array}$$

SOLUTION: This is Problem 5(a) in 6.2 (except we're just computing the transform of the solution). Using the Fourier cosine transform, we get

$$\overline{U}_t(\alpha,t) = -\alpha^2\overline{U}(\alpha,t) - \frac{2}{\pi}\cdot 0$$

From which we ultimately get:

$$\overline{U}(\alpha,t) = F(s)\,\mathrm{e}^{-\alpha^2 t}$$

12. Find the Fourier transform for the function $f(x) = 1$ if $-1 \le x \le 1$, and 0 elsewhere.

    SOLUTION: This is Example 1, section 6.3.

13. Find the Fourier transform of $f(x) = \mathrm{e}^{-c|x|}$, $c > 0$.

    SOLUTION: This is Example 2, section 6.3.

14. Find the Fourier transform of $f'(x)$ and $f''(x)$ in terms of the Fourier transform of $f(x)$.

    SOLUTION: These are very similar to the sine/cosine transform formulas. Let's try it:

$$\mathcal{F}(f'(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)\mathrm{e}^{-i\alpha x}\,dx \quad \Rightarrow \quad \begin{array}{ccc} + & \mathrm{e}^{-i\alpha x} & f'(x) \\ - & -i\alpha\mathrm{e}^{-i\alpha x} & f(x) \end{array}$$

$$\left(\frac{1}{\sqrt{2\pi}}f(x)\mathrm{e}^{-i\alpha x}\right|_{-\infty}^{\infty} + i\alpha\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-i\alpha x}\,dx$$

For the limit, we'll assume $f(x) \to 0$ as $x \to \pm\infty$. From this we get that the Fourier transform of $f'(x)$ is:

$$i\alpha F(\alpha)$$

Now for the second transform:

$$\mathcal{F}(f''(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f''(x)\mathrm{e}^{-i\alpha x}\,dx \quad \Rightarrow \quad \begin{array}{ccc} + & \mathrm{e}^{-i\alpha x} & f''(x) \\ - & -i\alpha\mathrm{e}^{-i\alpha x} & f'(x) \\ + & i^2\alpha^2\mathrm{e}^{-i\alpha x} & f(x) \end{array}$$

$$\frac{1}{\sqrt{2\pi}}\left[f'(x)\mathrm{e}^{-i\alpha x} + i\alpha f(x)\mathrm{e}^{-i\alpha x}\right|_{-\infty}^{\infty} + (i\alpha)^2\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-i\alpha x}\,dx$$

For the limit, we'll assume $f'(x)$ and $f(x)$ go to 0 as $x \to \pm\infty$. Then the transform of the second derivative is given by:

$$(i\alpha)^2F(\alpha)$$

And more generally, we see that

$$\mathcal{F}(f^{(n)}(x)) = (i\alpha)^nF(\alpha)$$

15. Show that $\mathcal{F}(f(x-c)) = e^{-ic\alpha}F(\alpha)$, where $F(\alpha)$ is the Fourier transform of $f(x)$.

SOLUTION: This is mainly a change of variables problem. Let's compare the two sides:

$$\mathcal{F}(f(x-c)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-c)e^{-i\alpha x}\, dx$$

and

$$e^{-ic\alpha}F(\alpha) = e^{-ic\alpha}\frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)e^{-i\alpha x}\, dx$$

To get from the top equation to the bottom equation, we need a change of variables- Let's try $z = x - c$, or $x = z + c$. Then starting again from the top,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-c)e^{-i\alpha x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z)e^{-i\alpha(z+c)}\, dz = e^{-ic\alpha}\frac{1}{2\pi} \int_{-\infty}^{\infty} f(z)e^{-i\alpha z}\, dz$$

And this last expression is exactly $e^{-ic\alpha}F(\alpha)$, which is what we wanted.

16. Find the Fourier transform of the solution for the heat equation below:

$$\begin{array}{l} u_t = 4u_{xx}, \quad -\infty < x < \infty, t > 0 \\ u(x,0) = f(x) \\ \lim_{|x|\to\infty} u(x,t) = 0 \end{array}$$

SOLUTION: See pages 242-243, and note that our solution will be

$$\overline{U}(\alpha, t) = F(s)e^{-4\alpha^2 t}$$

17. Find the Fourier transform of the solution for the heat equation below (the sides of the infinite rod are uninsulated):

$$\begin{array}{l} u_t = u_{xx} - u, \quad -\infty < x < \infty, t > 0 \\ u(x,0) = f(x) \\ \lim_{|x|\to\infty} u(x,t) = 0 \end{array}$$

SOLUTION: We do something similar to the previous problem.

$$\overline{U}_t(\alpha, t) = -\alpha^2\overline{U}(\alpha, t) - \overline{U}(\alpha, t) = -(1+\alpha^2)\overline{U}(\alpha, t)$$

Solving this and using our initial state, we get

$$\overline{U}(\alpha, t) = F(\alpha)e^{-(1+\alpha^2)t}$$