Summary of Eigenfunctions for Homog BCs (Corrected)

Given: $X'' + \lambda X = 0$ on the interval $[0, L]$ with the following sets of boundary conditions (BCs):

BC 1	BC 2	BC 3	BC 4
$X(0) = 0$	$X(0) = 0$	$X'(0) = 0$	$X'(0) = 0$
$X(L) = 0$	$X'(L) = 0$	$X(L) = 0$	$X'(L) = 0$

In each situation, we find the eigenvalues λ_n and the eigenfunctions $X_n(x)$.

BC 1: (Homogeneous Dirichlet) In this case, we obtain

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \text{ for } n = 1, 2, 3, \ldots$$

This leads to the Fourier sine series for $f(x)$ on $[0, L]$:

$$F(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{where } b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

BC 2: In this case, we obtain

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2 \qquad X_n(x) = \sin\left(\frac{(2n-1)\pi x}{2L}\right), \text{ for } n = 1, 2, 3, \ldots$$

This leads to the Fourier sine series (with frequencies at odd multiples of $\pi/2L$) for $f(x)$ on $[0, L]$:

$$F(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{(2n-1)\pi x}{2L}\right) \quad \text{where } b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{(2n-1)\pi x}{2L}\right)\,dx$$

BC 3: In this case, we obtain

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2 \qquad X_n(x) = \cos\left(\frac{(2n-1)\pi x}{2L}\right), \text{ for } n = 1, 2, 3, \ldots$$

This leads to the Fourier cosine series for $f(x)$ on $[0, L]$, and remember that $\lambda \neq 0$ in this case.

$$F(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{(2n-1)\pi x}{2L}\right) \quad \text{where } a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{(2n-1)\pi x}{2L}\right)\,dx$$

BC 4: (Homogeneous Neumann) In this case, we obtain

$$\lambda_0 = 0 \qquad X_0(x) = 1$$
$$\lambda_n = \left(\tfrac{n\pi}{L}\right)^2 \qquad X_n(x) = \cos\left(\tfrac{n\pi x}{L}\right), \text{ for } n = 1, 2, 3, \ldots$$

This leads to the Fourier cosine series for $f(x)$ on $[0, L]$:

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \quad \text{where } a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$$

Recall that in each case, eigenfunctions corresponding to distinct eigenvalues are orthogonal. Also recall the full Fourier series on the full interval $[-L, L]$:

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{with}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$