Problem Set 7 (4.1, 4.2)

Due: 4.1: 1(a), 2(c), 3(a) and 4.2: 1(c), 2(c)

4.1.1(a) Solve the heat equation if $L = \pi$, the two ends are held at 0 degrees, the initial temperature is uniformly 20 degrees.

$$\begin{aligned} u_t &= 2u_{xx}, \qquad 0 < x < \pi \\ u(x,0) &= 20 \\ u(0,t) &= 0 \\ u(\pi,t) &= 0 \end{aligned}$$

SOLUTION: Separate variables and get the eigenfunctions. Because we have $X(0) = 0$ and $X(\pi) = 0$, let's keep the constant 2 with T (like in the exam).

$$XT' = 2X''T \quad \Rightarrow \quad \frac{T'}{2T} = \frac{X''}{X} = -\lambda$$

Therefore,

$$\begin{array}{c} X'' + \lambda X = 0 \\ X(0) = X(\pi) = 0 \end{array} \qquad T' + 2\lambda T = 0$$

For the BVP in X, we know that $\lambda_n = n^2$ and $X_n(x) = \sin(nx)$.

Changing the DE in T now that we have λ,

$$T' + 2n^2T = 0 \quad \Rightarrow \quad T' = -2n^2T \quad \Rightarrow \quad T_n(t) = \mathrm{e}^{-2n^2t}$$

Putting our solution together,

$$u(x,t) = \sum_{n=1}^{\infty} b_n \mathrm{e}^{-2n^2t} \sin(nx)$$

That takes care of everything except for the initial condition:

$$u(x,0) = 20 = \sum_{n=1}^{\infty} b_n \sin(nx)$$

Therefore,

$$b_n = \frac{2}{\pi}\int_0^{\pi} 20\sin(nx)\,dx = \left(\frac{-40}{\pi n}\cos(nx)\right|_0^{\pi} = -\frac{40}{\pi n}((-1)^n - 1) = \begin{cases} 80/n\pi \text{ if } n \text{ odd} \\ 0 \text{ if } n \text{ even} \end{cases}$$

Using only the odd indices then $n = 2k - 1$, we get

$$u(x,t) = \frac{80}{\pi}\sum_{k=1}^{\infty} \mathrm{e}^{-2(2k-1)^2t}\frac{\sin((2k-1)x}{2k-1}$$

4.1.2(c) Solve the heat equation if $L = 2$, the two ends are insulated, the initial temperature is given by the piecewise function $g(x)$ below.

$$\begin{aligned} u_t &= 4u_{xx}, \qquad 0 < x < 2 \\ u(x,0) &= g(x) \\ u_x(0,t) &= 0 \\ u_x(2,t) &= 0 \end{aligned} \qquad g(x) = \begin{cases} 10 & \text{if } 0 \le x \le 1 \\ 0 & \text{if } 1 < x \le 2 \end{cases}$$

SOLUTION: Separate variables and get the eigenfunctions. Because we have $X'(0) = 0$ and $X'(2) = 0$, let's keep the constant 4 with T.

$$XT' = 4X''T \quad \Rightarrow \quad \frac{T'}{4T} = \frac{X''}{X} = -\lambda$$

Therefore,

$$\begin{array}{c} X'' + \lambda X = 0 \\ X'(0) = X'(2) = 0 \end{array} \qquad T' + 4\lambda T = 0$$

For the BVP in X, we know that we have two cases: $\lambda_0 = 0$ with $X_0(x) = 1$, and the more normal case $\lambda_n = (n\pi/2)^2$ and $X_n(x) = \cos(n\pi x/2)$.

Changing the DE in T now that we have λ (two cases):

$$\begin{array}{l} T' + 0 = 0 \\ T_0(t) = 1 \end{array} \qquad \begin{array}{l} T' + 4(n^2\pi^2/4)T = 0 \\ T' = -n^2\pi^2 T \\ T_n(t) = \mathrm{e}^{-n^2\pi^2 t} \end{array}$$

Putting our solution together,

$$u(x,t) = b_0 + \sum_{n=1}^{\infty} b_n \mathrm{e}^{-n^2\pi^2 t} \cos(n\pi x/2)$$

That takes care of everything except for the initial condition:

$$u(x,0) = g(x) = b_0 + \sum_{n=1}^{\infty} b_n \cos(n\pi x/2)$$

We compute as usual:

$$b_0 = \frac{a_0}{2} = \frac{1}{2}\frac{2}{2}\int_0^2 g(x)\cos(n\pi x/2)\,dx = \frac{1}{2}\int_0^1 10\,dx = 5$$

$$b_n = \frac{2}{2}\int_0^2 g(x)\cos(n\pi x/2)\,dx = 10\int_0^1 \cos(n\pi x/2)\,dx = \frac{20}{\pi n}\sin(n\pi/2)$$

We might try to "simplify" that expression more, but it's OK to leave it as is.

4.1.3(a) Solve the heat equation if $L = \pi$, the initial temperature is uniformly 100 degrees, with BCs given below.

$$\begin{aligned} u_t &= u_{xx}, \qquad 0 < x < \pi \\ u(x,0) &= 100 \\ u(0,t) &= 0 \\ u_x(\pi,t) &= 0 \end{aligned}$$

SOLUTION: Separate variables and get the eigenfunctions in X since $X(0) = 0$ and $X'(\pi) = 0$.

$$XT' = X''T \quad \Rightarrow \quad \frac{T'}{T} = \frac{X''}{X} = -\lambda$$

Therefore,

$$\begin{aligned} X'' + \lambda X &= 0 \\ X(0) &= 0 \qquad\qquad T' + \lambda T = 0 \\ X'(\pi) &= 0 \end{aligned}$$

For the BVP in X, we have BC2, so

$$\lambda_n = \left(\frac{(2n-1)}{2}\right)^2 \qquad \text{with} \qquad X_n(x) = \sin\left(\frac{(2n-1)x}{2}\right)$$

Now, $T_n(t) = \mathrm{e}^{-\lambda_n t}$, so that

$$u(x,t) = \sum_{n=1}^{\infty} d_n \mathrm{e}^{-(2n-1)^2 t/4} \sin\left(\frac{(2n-1)x}{2}\right)$$

Finally, $u_x(0,t) = 100$, so we can compute the d_n's:

$$100 = \sum_{n=1}^{\infty} d_n \sin\left(\frac{(2n-1)x}{2}\right)$$

Therefore,

$$d_n = 100\frac{2}{\pi}\int_0^{\pi} \sin\left(\frac{(2n-1)x}{2}\right)\,dx = -\frac{400}{(2n-1)\pi}(\cos((2n-1)\pi/2) - 1) = \frac{400}{(2n-1)\pi}$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{400}{(2n-1)\pi} \mathrm{e}^{-(2n-1)^2 t/4} \sin\left(\frac{(2n-1)x}{2}\right)$$

Section 4.2

4.2.1(c) Solve the wave equation with length $L = 2$, and the initial and boundary conditions shown (ends are nailed down).

$$\begin{aligned} u_{tt} &= 5u_{xx}, \qquad 0 < x < 2, t > 0 \\ u(x,0) &= 0 \\ u_t(x,0) &= 3 \\ u(0,t) &= 0 \\ u(2,t) &= 0 \end{aligned}$$

SOLUTION: Separate variables and get the eigenfunctions. Because we have $X(0) = 0$ and $X(2) = 0$, let's keep the constant 5 with T.

$$XT'' = 5X''T \quad \Rightarrow \quad \frac{T'}{5T} = \frac{X''}{X} = -\lambda$$

Therefore,

$$\begin{array}{cc} X'' + \lambda X = 0 & T'' + 5\lambda T = 0 \\ X(0) = 0 & T(0) = 0 \\ X(2) = 0 & T'(0) = 3 \\ & \text{(Special ICs here)} \end{array}$$

For the BVP in X, we know that $\lambda_n = n^2\pi^2/4$ and $X_n(x) = \sin(n\pi x/2)$.

Changing the DE in T now that we have λ,

$$T'' + \frac{5}{4}n^2\pi^2 T = 0 \quad \Rightarrow \quad r = \pm\frac{\sqrt{5}n\pi}{2}\,i = \pm\omega_n i$$

We see that $T_n(t) = c_n\cos(\omega_n t) + d_n\sin(\omega_n t)$. For $T_n(0) = 0$, we must have $c_n = 0$. For the other condition, we'll need the full solution.

Putting our solution together,

$$u(x,t) = \sum_{n=1}^{\infty}\sin(nx)\left[0\cdot\cos(\omega_n t) + d_n\sin(\omega_n t)\right] = \sum_{n=1}^{\infty} d_n\sin(nx)\sin(\omega_n t)$$

That takes care of everything except for the initial velocity:

$$u_t(x,0) = 3 = \sum_{n=1}^{\infty}\omega_n d_n\sin(nx) \quad \Rightarrow \quad \omega_n d_n = \frac{2}{2}\int_0^2 3\sin(nx)\,dx$$

Therefore,

$$d_n = \frac{3}{\omega_n}\cdot\frac{\cos(2n) - 1}{n}$$

These expressions don't simplify much, so we'll leave them as is.

4.2.2(c) Solve the wave equation.

$$\begin{aligned} u_{tt} &= 4u_{xx}, \qquad 0 < x < \pi, t > 0 \\ u(x,0) &= 1 \\ u_t(x,0) &= x \\ u_x(0,t) &= 0 \\ u_x(\pi,t) &= 0 \end{aligned}$$

SOLUTION: Separate variables and get the eigenfunctions. Because we have $X'(0) = 0$ and $X'(\pi) = 0$, let's keep the constant 4 with T.

$$XT'' = 4X''T \quad \Rightarrow \quad \frac{T''}{4T} = \frac{X''}{X} = -\lambda$$

Therefore,

$$\begin{array}{c} X'' + \lambda X = 0 \\ X'(0) = 0 \\ X'(\pi) = 0 \end{array} \qquad \begin{array}{c} T'' + 4\lambda T = 0 \\ \text{(Can't say anything else)} \end{array}$$

For the BVP in X, we know that $\lambda_n = n^2$ and $X_n(x) = \cos(nx)$.

In this case, we also have $\lambda_0 = 0$ with $X_0(x) = 1$.

Changing the DE in T now that we have two cases for λ,

$$\begin{array}{l} T'' = 0 \\ T_0(t) = c_0 + d_0 t \end{array} \qquad \begin{array}{l} T'' + 4n^2 T = 0 \\ r = \pm 2n\, i \\ T_n(t) = c_n \cos(2nt) + d_n \sin(2nt) \end{array}$$

The full solution is therefore below, and we go ahead and include the velocity:

$$u(x,t) = c_0 + d_0 t + \sum_{n=1}^{\infty} \cos(nx) \left[c_n \cos(2nt) + d_n \sin(2nt)\right]$$

$$u_t(x,t) = d_0 + \sum_{n=1}^{\infty} \cos(nx) \left[-2nc_n \sin(2nt) + 2nd_n \cos(2nt)\right]$$

Substitute in our initial conditions. First, $u(x,0) = 1$:

$$1 = c_0 + \sum_{n=1}^{\infty} c_n \cos(nx) \quad \Rightarrow \quad \begin{array}{l} c_0 = 1 \\ c_n = 0 \text{ for } n = 1, 2, \ldots \end{array}$$

Next is initial velocity, $u_t(x,0) = x$

$$x = d_0 + \sum_{n=1}^{\infty} 2nd_n \cos(nx) \quad \Rightarrow \quad d_0 = \frac{1}{\pi}\int_0^{\pi} x\, dx = \frac{\pi}{2}$$

The d_n terms don't simplify too much (use integration by parts to evaluate):

$$2nd_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\, dx \qquad \begin{array}{ccc} + & x & \cos(nx) \\ - & 1 & \sin(nx)/n \\ + & 0 & -\cos(nx)/n^2 \end{array}$$

Just the integral evaluates to:

$$\left(\frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right|_0^{\pi} = (0 + \frac{(-1)^n}{n^2}) - (0 + \frac{1}{n^2}) = \begin{cases} -2/n^2 & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

so if n is odd, $d_n = -2/(n^3\pi)$.

$$u(x,t) = 1 + \frac{\pi}{2}t - \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} \cos(nx)\sin(2nt)$$