Problem Set 9 (5.1, 5.2)

5.1: 1, 4, 8. 5.2: 5, 6, 9

5.1.1 (Solution in the back of the book)

5.1.4 Solve:

$$\begin{aligned} -2u_x + 6u_y &= 0 \\ u(4y, y) &= 2y + 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \xi &= x \\ \eta &= 6x + 2y \end{aligned} \quad \Rightarrow \quad \begin{aligned} u_x &= u_\xi \cdot 1 + u_\eta \cdot 6 \\ u_y &= 0 + u_\eta \cdot 2 \end{aligned}$$

$$-2u_x + 6u_y = 0 \quad \Rightarrow \quad u_\xi = 0 \quad \Rightarrow \quad u(\xi, \eta) = g(\eta) \quad \Rightarrow \quad u(x, y) = g(6x + 2y)$$

Using the "initial condition", we have:

$$u(4y, y) = g(26y) = 2y + 1 \quad \Rightarrow \quad g(z) = \frac{1}{13}z + 1$$

Overall then, the solution is

$$u(x, y) = \frac{1}{13}(6x + 2y) + 1$$

5.1.8 Solve:

$$\begin{aligned} u_x + 4u_y - 2u &= \mathrm{e}^{x+y} \\ u(x, 0) &= \cos(x) \end{aligned} \quad \Rightarrow \quad \begin{aligned} \xi &= x \\ \eta &= 4x - y \end{aligned} \quad \Rightarrow \quad \begin{aligned} u_x &= u_\xi \cdot 1 + u_\eta \cdot 4 \\ u_y &= 0 + u_\eta \cdot (-1) \end{aligned}$$

$$u_x + 4u_y - 2u = \mathrm{e}^{x+y} \quad \Rightarrow \quad u_\xi - 2u = \mathrm{e}^{\xi + (4\xi - \eta)} = \mathrm{e}^{5\xi}\mathrm{e}^{-\eta}$$

The integrating factor is $\mathrm{e}^{-2\xi}$, so that we get:

$$\left(\mathrm{e}^{-2\xi}u\right)_\xi = \mathrm{e}^{3\xi}\mathrm{e}^{-\eta} \quad \Rightarrow \quad \mathrm{e}^{-2\xi}u = \frac{1}{3}\mathrm{e}^{3\xi}\mathrm{e}^{-\eta} + g(\eta)$$

Therefore,

$$u(\xi, \eta) = \frac{1}{3}\mathrm{e}^{5\xi - \eta} + g(\eta)\mathrm{e}^{2\xi}$$

or, returning to x, y:

$$u(x, y) = \frac{1}{3}\mathrm{e}^{x+y} + g(4x - y)\mathrm{e}^{2x}$$

Using the "initial condition", we have:

$$u(x, 0) = \cos(x) = \frac{1}{3}\mathrm{e}^{x} + g(4x)\mathrm{e}^{2x}$$

Therefore,

$$g(4x) = \left(\cos(x) - \frac{1}{3}\mathrm{e}^{x}\right)\mathrm{e}^{-2x} \quad \Rightarrow \quad g(z) = \left(\cos(z/4) - \frac{1}{3}\mathrm{e}^{z/4}\right)\mathrm{e}^{-2z/4}$$

Overall then, the solution is

$$u(x, y) = \frac{1}{3}\mathrm{e}^{x+y} + g(4x - y)\mathrm{e}^{2x}$$

(It doesn't simplify by much if you want to substitute in for g, but this is fine).

5.2.5 Solve

$$\begin{aligned} u_x + 3x^2u_y - u &= 0 \\ u(2,y) &= 3y+1 \end{aligned} \quad \Rightarrow \quad \frac{dx}{1} = \frac{dy}{3x^2} \quad \Rightarrow \quad 3x^2\,dx = dy$$

Solving the ODE, we get $x^3 - y = C$, so our change of coordinates is given by

$$\begin{aligned} \xi &= x \\ \eta &= x^3 - y \end{aligned} \quad \Rightarrow \quad \begin{aligned} u_x &= u_\xi \cdot 1 + 3x^2 u_\eta \\ u_y &= 0 - u_\eta \end{aligned} \quad \Rightarrow \quad u_x + 3x^2 u_y = u_\xi$$

Therefore, the PDE becomes:

$$u_\xi - u = 0 \quad \Rightarrow \quad u(\xi,\eta) = g(\eta)\mathrm{e}^\xi \quad \Rightarrow \quad u(x,y) = g(x^3-y)\mathrm{e}^x$$

Using the initial condition,

$$u(2,y) = g(8-y)\mathrm{e}^2 = 3y+1 \quad \Rightarrow \quad g(8-y) = (3y+1)\mathrm{e}^{-2}$$

$$g(z) = (3(8-z)+1)\mathrm{e}^{-2} = (25-3z)\mathrm{e}^{-2}$$

Now, $u(x,y) = g(x^3-y)\mathrm{e}^{-x}$, so

$$u(x,y) = (25 - 3(x^3-y))\mathrm{e}^{x-2} = (-3x^3 + 3y + 25)\mathrm{e}^{x-2}$$

5.2.6 Solve

$$\begin{aligned} u_x + xu_y &= x^2y \\ u(x, 2x + x^2/2) &= 5x \end{aligned} \quad \Rightarrow \quad \frac{dx}{1} = \frac{dy}{x} \quad \Rightarrow \quad x\,dx = dy$$

Solving the ODE, we get $\frac{x^2}{2} - y = C$, so our change of coordinates is given by

$$\begin{aligned} \xi &= x \\ \eta &= x^2/2 - y \end{aligned} \quad \Rightarrow \quad \begin{aligned} u_x &= u_\xi + xu_\eta \\ u_y &= 0 - u_\eta \end{aligned} \quad \Rightarrow \quad u_x + xu_y = u_\xi$$

Therefore, the PDE becomes: $u_\xi = x^2y$, which we need to change into ξ, η. From the equation for η, we get:

$$\eta = x^2/2 - y \quad \Rightarrow \quad y = \frac{x^2}{2} - \eta = \frac{1}{2}\xi^2 - \eta$$

Therefore, x^2y becomes

$$\xi^2\left(\frac{1}{2}\xi^2 - \eta\right) \quad \Rightarrow \quad \frac{1}{2}\xi^4 - \xi^2\eta$$

The differential equation is:

$$u_\xi = \frac{1}{2}\xi^4 - \xi^2\eta \quad \Rightarrow \quad u = \frac{1}{10}\xi^5 - \frac{1}{3}\xi^3\eta + g(\eta)$$

Now the solution is:

$$u(x,y) = \frac{1}{10}x^5 - \frac{1}{3}x^3(x^2/2 - y) + g(x^2/2 - y) = -\frac{1}{15}x^5 + \frac{1}{3}x^3y + g(x^2/2-y)$$

Using the initial condition to determine g, we have

$$u(x, 2x + x^2/2) = 5x \quad \Rightarrow \quad \frac{1}{10}x^5 + \frac{2}{3}x^4 + g(-2x) = 5x$$

Therefore,

$$5x = -\frac{1}{8}x^4 + \frac{1}{2}x^2(2x + x^2) + g(-2x) = -\frac{3}{8}x^4 + x^3 + g(-2x)$$

And we have:

$$g(-2x) = \frac{3}{8}x^4 - x^3 + 5x \quad \Rightarrow \quad g(z) = \frac{3}{8}(-z/2)^4 - (-z/2)^3 + 5(-z/2)$$

This doesn't simplify much, so we'll leave it at that, and our solution is:

$$u(x, y) = \frac{1}{10}x^5 - \frac{1}{3}x^3(x^2 - y) + g(x^2/2 - y)$$

5.2.9 (Not graded)

$$\begin{aligned} &u_x + 2u_y + 3u_z = 0 \\ &u(x, y, 0) = xy \end{aligned}$$

Given $au_x + bu_y + cu_z$, we're told that we need to solve the system of ODEs:

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c}$$

We're also told to solve the first equation for $h_1(x, y)$:

$$\frac{dx}{a} = \frac{dy}{b} \quad \Rightarrow \quad bx - ay = C \quad \Rightarrow \quad h_1(x, y) = 2x - y$$

And

$$\frac{dx}{a} = \frac{dz}{c} \quad \Rightarrow \quad cx - az = C \quad \Rightarrow \quad h_2(x, y) = 3x - z$$

Now we have the change of coordinates: xi, eta and zeta:

$$\begin{aligned} \xi &= x \\ \eta &= 2x - y \\ \zeta &= 3x - z \end{aligned}$$

Let's see what happens to our derivatives now:

$$\begin{aligned} u_x &= u_\xi \xi_x + u_\eta \eta_x + u_\zeta \zeta_x \\ u_y &= u_\xi \xi_y + u_\eta \eta_y + u_\zeta \zeta_y \\ u_z &= u_\xi \xi_z + u_\eta \eta_z + u_\zeta \zeta_z \end{aligned} \quad \Rightarrow \quad \begin{aligned} u_x &= u_\xi + 2u_\eta + 3u_\zeta \\ u_y &= 0 - u_\eta + 0 \\ u_z &= 0 + 0 - u_\zeta \end{aligned}$$

Therefore, $u_x + 2u_y + 3u_z = u_\xi$, and the differential equation becomes:

$$u_\xi = 0 \quad \Rightarrow \quad u(\xi, \eta, \zeta) = g(\eta, \zeta)$$

Following along, we have the solution:

$$u(x, y, z) = g(2x - y, 3x - z)$$

Now, $u(x, y, 0) = g(2x - y, 3x) = xy$. If $w_1 = 2x - y$ and $w_2 = 3x$, then

$$g(w_1, w_2) = \frac{w_2}{3}(2x - w_1) = \frac{w_2}{3}(2\frac{w_2}{3} - w_1) = \frac{2}{9}w_2^2 - \frac{1}{3}w_2 w_1$$

Now we back substitute $2x - y = w_1, 3x - z = w_2$ to get

$$u(x, y, z) = -\frac{1}{3}(2x - y)(3x - z) + \frac{2}{9}(3x - z)^2$$

NOTE: The book uses $z - 3x$ instead of $3x - z$, and notice that they have:

$$u(x, y, z) = \frac{1}{3}(2x - y)(z - 3x) + \frac{2}{9}(z - 3x)^2$$

so our solutions are equivalent.