SOLUTIONS to the review

1. - Define "eigenvalue" (and eigenfunction) in the context of a linear operator L.

     SOLUTION: Let L be a linear operator. If there is a constant λ and non-zero function y so that $L(y) = \lambda y$, then λ is an eigenvalue, and y is the corresponding eigenfunction.

   - Given the definition, how is $y'' + \lambda y = 0$ an eigenvalue problem?

     SOLUTION: $L(y) = -y''$.

2. What does it mean for a PDE to be **well-posed**?

   SOLUTION: Three things- (i) A solution exists, (ii) The solution is unique, and (iii) The problem is stable.

3. Solve by using ODE methods: $u_{xy} = x - y$

   SOLUTION: Integrate with respect to y, then with respect to x:

   $$u_x = xy - \frac{1}{2}y^2 + f_1(x)$$

   where $f_1(x)$ is an arbitrary function of x. Integrate with respect to x:

   $$u(x, y) = \frac{1}{2}x^2 y - \frac{1}{2}xy^2 + \int f_1(x)\, dx + g(y)$$

4. Solve by using ODE methods: $u_y + xu = 2$

   SOLUTION: Think of this like a first order linear DE, but be sure to distinguish between the variables- In this case, y is the independent variable, so the x is treated as a constant. Therefore, the integrating factor is e^{xy}. Multiply both sides by that, and continue:

   $$\frac{\partial}{\partial y}\left(\mathrm{e}^{3xy}u\right) = 2\mathrm{e}^{xy} \Rightarrow \mathrm{e}^{xy}u = \frac{2}{x}\mathrm{e}^{xy} + g(x) \Rightarrow u(x, y) = \frac{2}{x} + g(x)\mathrm{e}^{-xy} \quad x \neq 0$$

5. Find all solutions to $u_y = 2x$ (using ODE methods) that also satisfies $u(x, 3) = \sin(x)$.

   SOLUTION: $u(x, y) = 2xy + f(x)$, so

   $$u(x, 3) = 6x + f(x) = \sin(x) \quad \Rightarrow \quad f(x) = \sin(x) - 6x$$

   Therefore, $u(x, y) = 2xy + \sin(x) - 6x$.

6. Suppose we're looking for product solutions to $u_x + u_y = 0$. Then we set $u = XY$, and with appropriate algebra, we get

   $$\frac{X'}{X} = -\frac{Y'}{Y}$$

What is the justification in setting these equal to a constant?

SOLUTION: We're thinking that $f(x) = g(y)$, for all x and y. In particular, the equation must be true for $y = y^*$, or $f(x) = g(y^*) = \lambda$. (And in a similar way, $g(y) = f(x^*) = \lambda$). Therefore, both functions must be equal to the same constant.

7. Suppose we are asked to solve the eigenvalue problem

$$\begin{aligned} y'' + \lambda y &= 0 \\ y(a) = y(b) &= 0 \end{aligned}$$

(a) Suppose we want to change the variable from $a < x < b$ to $z = \frac{L}{b-a}(x-a)$.

i. Justify: $\dfrac{dy}{dx} = \dfrac{dy}{dz}\dfrac{dz}{dx}$
SOLUTION: This is the chain rule. y will be a function of z, and z is a function of x.

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{L}{b-a}$$

ii. Write the BVP in terms of y, z:
SOLUTION: It's important to keep track of variables. The original equation is with respect to x, so by using our substitutions:

$$\frac{d^2y}{dz^2}\frac{L^2}{(b-a)^2} + \lambda y = 0 \quad \Rightarrow \quad \frac{d^2y}{dz^2} + \omega y = 0$$

with homogeneous boundary conditions in z: $y(0) = y(1) = 0$.

(b) Notice that we changed the interval so that $0 < z < L$. Use this technique to solve $y'' + \lambda y = 0$ with $y(-1) = y(1) = 0$.
SOLUTION: With the substitution $z = \frac{1}{2}(x+1)$, our interval changes from $x \in [-1, 1]$ to $z \in [0, 1]$. We'll keep the original λ in place this time to get

$$\frac{d^2y}{dz^2} + 4\lambda y = 0 \quad \Rightarrow \quad y'' + \omega y = 0$$

with $y(0) = y(1) = 0$. We know the eigenvalues and eigenfunctions:

$$\omega_n = n^2\pi^2 \qquad y_n(z) = \sin(n\pi z)$$

Conversion back to λ_n, x will give:

$$\lambda_n = \frac{n^2\pi^2}{4} \qquad y_n(x) = \sin\left(\frac{n\pi}{2}(x+1)\right)$$

8. Is the problem well posed? (You may decide in terms of existence and uniqueness only):

$$y'' + y' - 2y = 0, \quad y(0) = 0, y'(1) = 0, 0 \le x \le 1$$

SOLUTION: Use the characteristic equation: $r^2 + r - 2 = 0$, which gives $r = -2, 1$, so that

$$y(x) = C_1 e^{-2x} + C_2 e^{x}$$

Using the boundary conditions, we should see that there is a unique solution (using Cramer's rule, for instance, or by looking at the determinant of the coefficients). Therefore, this problem is well posed.

9. For each operator below, either prove that it is linear, or show that it is not linear:

   (a) $L(u) = yu_x - x^2 u_y + 2u$

   SOLUTION: Each of the individual terms is linear, so we expect the sum to be linear as well:

   $$L(c_1u_1 + c_2u_2) = y(c_1u_1 + c_2u_2)_y - x^2(c_1u_1 + c_2u_2)_x + 2(c_1u_1 + c_2u_2)$$

   Therefore, break everything out before we combine by factoring C_1, C_2:

   $$= c_1y(u_1)_y + c_2y(u_2)_y - c_1x^2(u_1)_x - c_2x^2(u_2)_x + c_1 2u_1 + c_2 2u_2$$

   $$= c_1(y(u_1)_y - x^2(u_1)_x + 2u_1) + c_2(y(u_2)_y - x^2(u_2)_x + 2u_2) = c_1L(u_1) + c_2L(u_2)$$

   (b) $L(u) = u - u_{xxt} + uu_{tt}$

   SOLUTION: We suspect this will not be linear because of the uu_{tt} term, so we only check that $cL(u) = L(cu)$:

   $$cL(u) = cu - cu_{xxt} + cuu_{ttt}$$

   $$L(cu) = cu - cu_{xxt} + (cu)(cu_{tt}) \neq cL(u)$$

10. Suppose that u_1 solves $u_t - u_{xx} = f(x,t)$ and u_2 solves $u_t - u_{xx} = g(x,t)$ for some functions f, g. Find a solution u_3 that will solve the equation $u_t - u_{xx} = 5f(x,t) - 7g(x,t)$.

    SOLUTION: Think about the linearity of the operator. For example, if $L(u) = f$, then $L(2u) = 2f$. So in this case, we see that

    $$L(5u_1 - 7u_2) = 5L(u_1) - 7L(u_2) = 5f(x,t) - 7g(x,t).$$

11. Classify the heat equation, the wave equation and Laplace's equation as **hyperbolic**, **parabolic**, or **elliptic** using the definition (that is, show your work).

    SOLUTION: You should show that the heat equation is parabolic, the wave equation is hyperbolic, and Laplace's equation is elliptic (see page 43 of the text for more info).

12. Use ODE techniques to find the general solution of the following, where $u = u(x, y)$:

$$yu_{xy} + 2u_x = x$$

Hint: The equation can be expressed first as $(...)_x = x$.

SOLUTION: From the hint,

$$(yu_y + 2u)_x = x \quad \Rightarrow \quad yu_y + u = \frac{1}{2}x^2 + f(y)$$

for some arbitrary function f. Divide by y to get a first order linear DE in $u(y)$:

$$u_y + \frac{2}{y}u = \frac{x^2}{2y} + \frac{f(y)}{y}$$

We see the that the integrating factor is

$$\mathrm{e}^{\int 2/y\,dy} = \mathrm{e}^{2\ln(y)} = y^2$$

Multiply both sides, and re-write the left side:

$$(y^2u)_y = \frac{x^2}{2}y + yf(y)$$

Antidifferentiate with respect to y:

$$y^2u = \frac{1}{4}x^2y^2 + F(y) + g(x) \qquad \text{where } F(y) = \int yf(y)\,dy$$

Finally, solve for u:

$$u(x, y) = \frac{1}{4}x^2 + \frac{F(y) + g(x)}{y^2} \qquad \text{for arbitrary } F, g$$

13. Given your solution to the previous problem, find a particular solution to the boundary conditions:

$$u(x, 1) = 0 \qquad u(0, y) = 0$$

SOLUTION: For the first constraint, we get

$$u(x, 1) = \frac{1}{4}x^2 + F(1) + g(x) = 0 \quad \Rightarrow \quad g(x) = -\frac{1}{4}x^2 - F(1)$$

From the second constraint,

$$u(0, y) = \frac{F(y) + g(0)}{y^2} = 0 \quad \Rightarrow \quad F(y) = g(0) = C$$

Put these together (with $F(1) = C$ as well), to get

$$u(x, y) = \frac{1}{4}x^2 + \frac{1}{y^2}\left(C - \frac{1}{4}x^2 - C\right) = \frac{1}{4}x^2 - \frac{x^2}{4y^2}$$

14. Solve the following PDE using separation of variables. You do not need to justify your solutions to the underlying eigenvalue problem.

$$\begin{array}{l} u_t = u_{xx} \quad 0 < x < 2, t > 0 \\ u(0,t) = 0 \\ u(2,t) = 0 \\ u(x,0) = 3\sin(\pi x) - 4\sin((3\pi/2)x), 0 < x < 2 \end{array}$$

SOLUTION: Break out the two ODEs as usual:

$$XT' = X''T \quad \Rightarrow \quad \frac{T'}{T} = \frac{X''}{X} = -\lambda$$

so that

$$T' + \lambda T = 0 \qquad X'' + \lambda X = 0 \text{ with } X(0) = X(2) = 0$$

The eigenvalue problem in X is using BC 1, so we get (for $X(x)$)

$$\lambda_n = \frac{n^2\pi^2}{4} \qquad X_n(x) = \sin\left(\frac{n\pi x}{2}\right)$$

Now that λ_n is set, we can also solve for T:

$$T_n(t) = \mathrm{e}^{-\lambda_n t} = \mathrm{e}^{-n^2\pi^2 t/4}$$

The full solution so far is:

$$u(x,t) = \sum_{n=1}^{\infty} C_n \mathrm{e}^{-n^2\pi^2 t/4} \sin\left(\frac{n\pi x}{2}\right)$$

To solve for the initial condition, by observation we see $C_2 = 3$ and $C_3 = -4$, so that the solution is:

$$u(x,t) = 3\mathrm{e}^{-\pi^2 t} \sin(\pi x) - 4\mathrm{e}^{-9\pi^2 t/4} \sin\left(\frac{3\pi x}{2}\right)$$

15. Separate the PDE into a system of ODEs (you do NOT need to solve each ODE, just set them up).

(a) $u_{xx} - xu_y + xu = 0$

$$\frac{X''}{xX} = \lambda \qquad \Rightarrow \quad X'' - \lambda x X = 0$$

$$fracY'Y + 1 = \lambda \quad \Rightarrow \quad Y' + (1 - \lambda Y = 0$$

(b) $u_t = u_{xx} + u_{yy}$

SOLUTION: First remember that $u(x,y,t) = XYT$ this time around, so we get

$$XYT' = X''YT + XY''T \quad \Rightarrow \quad \frac{T'}{T} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda_1$$

so that

$$T' + \lambda_1 T = 0 \text{ and } \frac{X''}{X} = -\frac{Y''}{Y} - \lambda_1 = -\lambda_2$$

In X and Y we then get

$$X'' + \lambda_2 X = 0 \qquad \text{and} \qquad Y'' + (\lambda_1 - \lambda_2)Y = 0$$

16. Solve the PDE by using separation of variables: $y^2 u_x + x^2 u_y = 0$

SOLUTION: Using our usual technique, we should get

$$\frac{1}{x^2}\frac{X''}{X} = -\frac{1}{y^2}\frac{Y'}{Y} = -\lambda$$

We construct the two ODEs:

$$\frac{dX}{dx} = -\lambda x^2 X \qquad \frac{dY}{dy} = \lambda y^2 Y$$

Now these are each separable as well (easier than linear?):

$$\int \frac{1}{X}\, dX = -\int \lambda x^2\, dx \quad \Rightarrow \quad \ln|X| = -\lambda x^3/3 + C_1$$

and

$$\int \frac{1}{Y}\, dY = \int \lambda y^2\, dy \quad \Rightarrow \quad \ln|Y| = \lambda y^3/3 + C_2$$

Exponentiate, then multiply together to get:

$$u_\lambda(x,y) = C\mathrm{e}^{(\lambda/3)(y^3 - x^3)}$$

17. Solve the following eigenvalue problem: $y'' + 2y' + (\lambda + 1)y = 0$, with $y(0) = 0$ and $y(\pi) = 0$.

SOLUTION: $r^2 + 2r + (\lambda + 1) = 0$, so $(r^2 + 2r + 1) + \lambda = 0$, or

$$r = -1 \pm \sqrt{-\lambda}$$

Three cases for the discriminant:

- Case 1: $\lambda < 0$, so $r = r_1, r_2$ distinct real numbers, and $y(x) = C_1\mathrm{e}^{r_1 x} + C_2\mathrm{e}^{r_2 x}$. Putting in the initial conditions:

$$\begin{aligned} C_1 + C_2 &= 0 \\ C_1\mathrm{e}^{\pi r_1} + C_2\mathrm{e}^{\pi r_2} &= 0 \end{aligned} \qquad \det\begin{vmatrix} 1 & 1 \\ \mathrm{e}^{\pi r_1} & \mathrm{e}^{\pi r_2} \end{vmatrix} = \mathrm{e}^{\pi r_1} - \mathrm{e}^{\pi r_2} \neq 0$$

Since the determinant is not zero, the only solution to the system of equations is $C_1 = C_2 = 0$.

- Case 2: $\lambda = 0$, so $r = -1, -1$ and $y(x) = \mathrm{e}^{-x}(C_1 + C_2 x)$
  Putting in the initial conditions leads to $C_1 = 0$ and $C_2 = 0$.
- Case 3: $\lambda > 0$, so $r = -1 \pm \sqrt{\lambda}\, i = -1 \pm \beta i$
  Now the solution is $y(x) = \mathrm{e}^{-x}\left(C_1 \cos(\beta x) + \sin(\beta x)\right)$. The first initial condition forces $C_1 = 0$, the second condition:

$$C_2 \mathrm{e}^{-\pi} \sin(\beta\pi) = 0$$

  so that $\sin(\beta\pi) = 0$ when $\beta = 1, 2, 3, \ldots$, or when $\lambda_n = n^2$ and $y_n(x) = \sin(nx)$.

18. (a) Given the eigenvalue problem $y'' + \lambda y = 0$ with $y(0) = 0$ and $y(L) = 0$, and if we are told that λ_1, λ_2 are two distinct eigenvalue with associated eigenfunctions y_1, y_2, then show that

$$(\lambda_1 - \lambda_2) \int_0^L y_1 y_2 \, dx = \int_0^L (y_1 y_2'' - y_1'' y_2) \, dx$$

SOLUTION: Since λ_1, y_1 are an eigenvalue/eigenvector pair for $y'' + \lambda y = 0$, then $L(y_1) = \lambda_1 y_1$, or

$$-y_1'' = \lambda_1 y_1 \qquad \text{and } -y_2'' = \lambda_2 y_2$$

Making these substitutions gives you the integral on the right.

(b) Use integration by parts to show

$$\int_0^L y_1 y_2'' \, dx = y_1 y_2' \big|_0^L - \int_0^L y_1' y_2' \, dx$$

Using a table,

sign	u	dv
$+$	y_1	y_2''
$-$	y_1'	y_2'

we get the expression to the right.

19. The **steady state** or **equilibrium** solution to the heat equation is a solution that does not depend on time (that is, $u = u(x)$).

(a) Find all the steady state solutions to the heat equation if the left end of the rod is held at 3 degrees, and the right end is held at 10 degrees.

SOLUTION: The dimensions of the rod are not given, so we'll assume that $0 < x < L$. Therefore, we solve:

$$u_{xx} = 0 \qquad \begin{array}{l} u(0) = 3 \\ u(L) = 10 \end{array} \quad \Rightarrow \quad u(x) = C_1 x + C_2 \quad \Rightarrow \quad u(x) = \frac{7}{L} x + 3$$

(b) Repeat the previous problem, but the left end is insulated and the right end is held at 10 degrees.

SOLUTION: We still get $u(x) = C_1 x + C_2$, so $u'(x) = C_1$. If we want that to be zero, $C_1 = 0$ and C_2 will be 10, or $u(x) = 10$.

20. Given the heat equation with nonhomogeneous boundary conditions:

$$\begin{aligned} &u_t = u_{xx} \\ &u(x,0) = f(x) \\ &u(0,t) = 10 \\ &u(5,t) = 30 \end{aligned}$$

(a) Find an equilibrium solution (call it $v(x)$ rather than $u(x)$) that satisfies the nonhomogeneous boundary conditions.

SOLUTION: We want $v(0) = 10$ and $v(5) = 30$, so like the previous problem, we get

$$v(x) = C_1 x + C_2 \quad \Rightarrow v(x) = 4x + 10$$

(b) Define $w(x,t) = u(x,t) - v(x)$. Write the heat equation in terms of w rather than u. Verify that w solves the heat equation with homogeneous boundary conditions.

SOLUTION: We note that

$$w_t = u_t + 0 \qquad w_{xx} = u_{xx} + 0$$

(the second zero is because $v''(x) = 0$) so that if u satisfies the heat equation, so does w.

For the initial and boundary conditions,

$$w(x,0) = f(x) - 4x - 10$$

with

$$w(0,t) = u(0,t) - v(0) = 10 - 10 = 0 \qquad w(5,t) = u(5,t) - v(5) = 30 - 30 = 0$$

(c) Remark: Using this, the overall original solution is $u(x,t) = w(x,t) + v(x)$.

21. Solve the heat equation intial-boundary-value problem

$$\begin{aligned} &u_t = u_{xx} \\ &u(x,0) = 3 + \cos(2\pi x) \\ &u_x(0,t) = 0 \\ &u_x(3,t) = 0 \end{aligned}$$

You do not need to justify your solutions to the underlying eigenvalue problem.

SOLUTION: You should find that

$$u(x,t) = 3 + \mathrm{e}^{-4\pi^2 t}\cos(2\pi x)$$

22. Solve the wave equation intial-boundary-value problem

$$\begin{aligned}&u_{tt} = u_{xx}\\&u(x,0) = 5\sin(2x) - 7\sin(4x)\\&u_t(x,0) = 0\\&u(0,t) = u(\pi,t) = 0\end{aligned}$$

You do not need to justify your solutions to the underlying eigenvalue problem.

SOLUTION: You should find that

$$u(x,t) = 5\sin(2x)\cos(2t) - 7\sin(4x)\cos(4t)$$

23. Solve Laplace's equation:

$$\begin{aligned}&u_{xx} + u_{yy} = 0\\&u(x,0) = \sin(3x)\\&u(x,1) = \sin(x)\\&u(0,y) = u(\pi,y) = 0\end{aligned}$$

You do not need to justify your solutions to the underlying eigenvalue problem.

SOLUTION:

$$X''Y + XY'' = 0 \quad \Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

Therefore, we have the pair of ODEs:

$$\begin{aligned}&X'' + \lambda X = 0\\&X(0) = X(\pi) = 0\end{aligned} \qquad \begin{aligned}&Y'' - \lambda Y = 0\\&X(x)Y(0) = \sin(3x)\\&X(x)Y(1) = \sin(x)\end{aligned}$$

From the left equations, we get $\lambda_n = n^2$ for $n = 1, 2, 3, \ldots$, with $X_n(x) = \sin(nx)$. Now that $\lambda_n > 0$, that fixes the solutions for $Y(y)$

$$r^2 - \lambda_n = 0 \quad \Rightarrow \quad r = \pm\sqrt{\lambda_n} = \pm n$$

Therefore, we have

$$Y_{n_1}(y) = \cosh(ny) \qquad Y_{n_2}(y) = \sinh(ny)$$

Our solutions are of the form:

$$u(x,t) = \sum_{n=1}^{\infty} C_n \cosh(ny)\sin(nx) + \sum_{n=1}^{\infty} D_n \sinh(ny)\sin(nx)$$

Now we look at the other boundary conditions:

$$u(x,0) = \sin(3x) = \sum_{n=1}^{\infty} C_n \sin(nx) + 0 \quad \Rightarrow \quad C_3 = 1$$

and all other $C_n = 0$. Checking the other boundary condition:

$$u(x,1) = \cosh(3)\sin(3x) + \sum_{n=1}^{\infty} D_n \sinh(n)\sin(nx)$$

From the left, we'll need $n = 1$ for $\sin(x)$ term, but we'll also need $n = 3$ to cancel out the $\sin(3x)$ terms. Therefore, we can reduce the problem by setting all other $D_n = 0$ so that

$$u(x,1) = \cosh(3)\sin(3x) + D_1 \sinh(1)\sin(x) + D_3 \sinh(3)\sin(3x)$$

We want the sum of the coefficients with $\sin(3x)$ to be 0:

$$\cosh(3) + D_3 \sinh(3) = 0 \qquad \Rightarrow \quad D_3 = -\frac{\cosh(3)}{\sinh(3)}$$

and the coefficient in front of $\sin(x)$ to be 1:

$$D_1 \sinh(1) = 1 \qquad \Rightarrow \quad D_1 = \frac{1}{\sinh(1)}$$

Now we have our full solution:

$$u(x,y) = \cosh(3y)\sin(3x) - \frac{\cosh(3)}{\sinh(3)}\sinh(3y)\sin(3x) + \frac{1}{\sinh(1)}\sinh(y)\sin(x)$$