Homework: Chapter 8 (Solutions)

1. For each of the following problems, find an appropriate function w that satisfies the boundary conditions, then let $u = v + w$ and convert the PDE to a PDE in v. Do not solve the PDE.

   (a)

$$\begin{array}{ll} \text{PDE} & u_t = ku_{xx} + x \quad 0 < x < L \\ \text{BCs} & u_x(0,t) = 1, \quad u(L,t) = t \end{array}$$

   SOLUTION: Let $w = x + t - L$, and $u = v + w$. Then the PDE in v is given by:

$$\begin{array}{ll} \text{PDE} & v_t = kv_{xx} + x - 1 \quad 0 < x < L \\ \text{BCs} & v_x(0,t) = 0, \quad v(L,t) = 0 \end{array}$$

   (b)

$$\begin{array}{ll} \text{PDE} & u_t = ku_{xx} + x \quad 0 < x < L \\ \text{BCs} & u_x(0,t) = t, \quad u_x(L,t) = t^2 \end{array}$$

   SOLUTION: Let $u = v + w$, where

$$w(x,t) = \frac{1}{2L}(t^2 - t)x^2 + xt$$

   Then the PDE in v is given by:

$$\begin{array}{ll} \text{PDE} & v_t = kv_{xx} + x - \left(\left(\frac{2t-1}{2L}\right) x^2 + x - k\frac{t^2-t}{L} \right) \quad 0 < x < L \\ \text{BCs} & v_x(0,t) = 0, \quad v(L,t) = 0 \end{array}$$

   (c)

$$\begin{array}{ll} \text{PDE} & u_{tt} = c^2 u_{xx} + xt \quad 0 < x < L \\ \text{BCs} & u(0,t) = 1, \quad u(L,t) = t \end{array}$$

   SOLUTION: Let $u = v + w$, where

$$w(x,t) = \frac{t-1}{L}x + 1$$

   Then the PDE in v is given by:

$$\begin{array}{ll} \text{PDE} & v_{tt} = c^2 v_{xx} + xt \quad 0 < x < L \\ \text{BCs} & v(0,t) = 0, \quad v(L,t) = 0 \end{array}$$

   (d)

$$\begin{array}{ll} \text{PDE} & u_{tt} = c^2 u_{xx} + xt \quad 0 < x < L \\ \text{BCs} & u_x(0,t) = 0, \quad u_x(L,t) = 1 \end{array}$$

SOLUTION: Let $u = v + w$, where

$$w(x,t) = \frac{1}{2L}x^2$$

Then the PDE in v is given by:

$$\begin{array}{ll} \text{PDE} & v_{tt} = c^2 v_{xx} + \frac{c^2}{L} + xt \quad 0 < x < L \\ \text{BCs} & v_x(0,t) = 0, \quad v_x(L,t) = 0 \end{array}$$

2. Solve 1(a) and (b) if the initial condition is $u(x,0) = f(x)$.

1(a) Continuing from where we left off,

SOLUTION: Let $w = x + t - L$, and $u = v + w$. Then the PDE in v is given by:

$$\begin{array}{ll} \text{PDE} & v_t = k v_{xx} + x - 1 \quad 0 < x < L \\ \text{BCs} & v_x(0,t) = 0, \quad v(L,t) = 0 \end{array}$$

You should find that the eigenvalues/functions are

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2 \qquad X_n = \cos\left(\frac{(2n-1)\pi x}{2L}\right)$$

If we write $kv_{xx} + x - 1$ as $kv_{xx} + Q(x,t)$, then we want to expand Q in terms of our eigenfunction basis:

$$-1 + x = \sum_{n=1}^{\infty} q_n \cos\left(\frac{(2n-1)\pi x}{2L}\right) \qquad q_n = \frac{2}{L}\int_0^L (x-1)\cos\left(\frac{(2n-1)\pi x}{2L}\right)\,dx$$

Furthermore, we'll need an initial condition for the ODEs:

$$v(x,0) = f(x) - x + L = \sum_{n=1}^{\infty} v_n(0)\cos\left(\frac{(2n-1)\pi x}{2L}\right)$$

or:

$$v_n(0) = \frac{2}{L}\int_0^L (f(x) - x - L)\cos\left(\frac{(2n-1)\pi x}{2L}\right)\,dx$$

We've computed all the pieces- Use the ansatz for $v(x,t)$ (as a series), and we get the system of ODEs:

$$v_n'(t) + k\lambda_n v_n(t) = q_n \quad \Rightarrow \quad v_n(t) = \frac{q_n}{\lambda_n} + C\mathrm{e}^{-k\lambda t}$$

where $C = v_n(0) - q_n/\lambda_n$

1(b) Continuing where we left off:

SOLUTION: Let $u = v + w$, where

$$w(x,t) = \frac{1}{2L}(t^2 - t)x^2 + xt$$

Then the PDE in v is given by:

PDE	$v_t = kv_{xx} + x - \left(\left(\frac{2t-1}{2L}\right)x^2 + x - k\frac{t^2-t}{L}\right)$	$0 < x < L$
BCs	$v_x(0,t) = 0, \quad v(L,t) = 0$	

The eigenvalues/functions are:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad X_n = \cos\left(\frac{n\pi}{L}\right)$$

(I am basically doing a copy/paste from the previous problem- You may leave your answer in general form unless otherwise requested)

If we write $kv_{xx} + ...$ as $kv_{xx} + Q(x,t)$, then we want to expand Q in terms of our eigenfunction basis:

$$Q(x,t) = \sum_{n=0}^{\infty} q_n \cos\left(\frac{n\pi x}{L}\right) \qquad \begin{cases} q_0 = \frac{1}{L}\int_0^L Q(x,t)\,dx \\ q_n = \frac{2}{L}\int_0^L Q(x,t)\cos\left(\frac{n\pi x}{L}\right)\,dx \end{cases}$$

Furthermore, we'll need an initial condition for the ODEs:

$$v(x,0) = f(x) - 0 = \sum_{n=0}^{\infty} v_n(0)\cos\left(\frac{n\pi x}{L}\right)$$

or:

$$\begin{cases} v_0(0) = \frac{1}{L}\int_0^L f(x)\,dx \\ v_n(0) = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \end{cases}$$

We've computed all the pieces- Use the ansatz for $v(x,t)$ (as a series), and we get the system of ODEs:

$$v_n'(t) + k\lambda_n v_n(t) = q_n \quad \Rightarrow \quad v_n(t) = \frac{q_n}{\lambda_n} + Ce^{-k\lambda t}$$

where $C = v_n(0) - q_n/\lambda_n$

3. Solve:

PDE	$u_t = ku_{xx} + e^{-t}$	$0 < x < \pi$
BCs	$u_x(0,t) = 0, \quad u_x(\pi,t) = 0$	
IC	$u(x,0) = \cos(2x)$	

SOLUTION: As before, I can almost just copy/paste the previous solution. Here's a summary of the solution. The eigenvalues/functions are:

$$\lambda_n = n^2, \quad n = 0, 1, 2, \ldots \qquad\qquad X_n = \cos(nx)$$

On the right side, we now have $Q(x,t)$ as just a function of t, which is constant with respect to the expansion in x. The first term of the cosine expansion is also a constant- Therefore,

$$Q(x,t) = \mathrm{e}^{-t} + \sum_{n=1}^{\infty} 0\cos(nx)$$

where we emphasize that all of the coefficients q_n are zero (except the first).

For the initial condition,

$$u(x,0) = \cos(2x) = \sum_{n=0}^{\infty} A_n(0)\cos(nx)$$

or $A_2(0) = 1$ and the rest are zero.

We've computed all the pieces- Use the ansatz for $u(x,t)$ (as a series), and we get the system of ODEs- We first write this in general form, then we'll break it out:

$$A_n'(t) + k\lambda_n A_n(t) = q_n$$

Now, for $n = 0$, we have:

$$A_0'(t) = \mathrm{e}^{-t} \quad \Rightarrow \quad A_0(t) = -\mathrm{e}^{-t} + C \quad \Rightarrow \quad A_0(t) = 1 - \mathrm{e}^{-t}$$

For $n = 2$, we have:

$$A_2'(t) + 4kA_2(t) = 0 \qquad A_2(0) = 1 \quad \Rightarrow \quad A_2(t) = \mathrm{e}^{-4kt}$$

All of the remaining ODEs will have zero as the solution. Therefore, the solution to this heat equation problem is:

$$u(x,t) = 1 - \mathrm{e}^{-t} + \mathrm{e}^{-4kt}\cos(2x)$$