Homework Set 1 SOLUTIONS (without turned in problems)

1. Partial differential equations can be classified many ways. A second order **linear** PDE in two variables is any function $u(x, y)$ whose PDE can be expressed as:

$$Au_{xx} + Bu_{xy} + CU_{yy} + Du_x + Eu_y + Fu = G$$

where A, B, C, D, E, F, G can be constants or functions of x, y. For example,

$$u_{tt} = \mathrm{e}^{-t} u_x + \sin(t)$$

is a linear second order DE. Furthermore, if $G = 0$, the equation is called **homogeneous**. For each equation below, state the order of the DE, state whether the PDE is linear or nonlinear, and whether it is homogeneous or not homogeneous:

(a) $u_t = u_x^2 + 2u_x + u$ SOLUTION: Not linear, it is homogeneous.

(b) $u_t = u_{xx} + 2u_x + u$ SOLUTION: Linear and homogeneous.

(c) $u_{tt} = uu_{xxxx} + \mathrm{e}^{-t}$ SOLUTION: Not linear and not homogeneous.

2. How many solutions can you find to the PDE $u_t = u_{xx}$? You might try a couple different approaches: (i) Assume u is a function of one variable only, and (ii) Assume $u = \mathrm{e}^{ax+bt}$.

SOLUTION:

- For (i), if u is a function of t alone, then $u_t = 0$ and $u(t) = C$. If u is a function of x alone, then $u'' = 0$, so $u(x) = C_1 x + C_2$.
- For (ii) (also done in class)

$$u_t = b\mathrm{e}^{bt}\mathrm{e}^{ax} \qquad u_x = a\mathrm{e}^{ax}\mathrm{e}^{bt} \qquad u_{xx} = a^2\mathrm{e}^{ax}\mathrm{e}^{bt}$$

Therefore, we require any a, b such that $a^2 = b$.

3. If $u_1(x, y)$ and $u_2(x, y)$ each satisfy the DE:

$$Au_{xx} + Bu_{xy} + CU_{yy} + Du_x + Eu_y + Fu = G$$

Then is it true that the sum satisfies it as well?

SOLUTION: If u_1 solves the DE, then

$$A(u_1)_{xx} + B(u_1)_{xy} + C(u_1)_{yy} + D(u_1)_x + E(u_1)_y + Fu_1 = G$$

And

$$A(u_2)_{xx} + B(u_2)_{xy} + C(u_2)_{yy} + D(u_2)_x + E(u_2)_y + Fu_2 = G$$

Therefore,

$$A(u_1+u_2)_{xx} + B(u_1+u_2)_{xy} + C(u_1+u_2)_{yy} + D(u_1+u_2)_x + E(u_1+u_2)_y + F(u_1+u_2) = G + G = 2G$$

Therefore, $u_1 + u_2$ would not solve the PDE unless $G = 0$ (unless the PDE is homogeneous).

4. **Review Question, Calc 2**: Compute the following integrals

SOLUTIONS:

$$\int \cos(t)\, dt = \sin(t) + C$$

$$\int \cos^2(t)\, dt = \frac{1}{2} \int (1 + \cos(2t))\, dt = \frac{1}{2}\left(t + \frac{1}{2}\sin(2t)\right) + C = \frac{1}{2}t + \frac{1}{4}\sin(2t) + C$$

$$\int \cos^3(t)\,dt = \int (\cos^2(t))\cos(t)\,dt \Rightarrow \quad \begin{array}{l} u = \sin(t) \\ \cos^2(t) = 1 - \sin^2(t) = 1 - u^2 \\ du = \cos(t)\,dt \end{array} \quad \Rightarrow \int 1 - u^2\,du$$

(and proceed as usual to get $\sin(t) - \frac{1}{3}\sin^3(t) + C$. Similarly, the last integral is also computed by a u, du substitution (I use $u = \sin(t)$, so $du = \cos(t)\,dt$ and:

$$\int \cos(t)\sin(t)\,dt = \int u\,du = \frac{1}{2}\sin^2(t) + C$$

5. **Review Question, Calc 3**: Recall that the line integral can be expressed a couple of different ways, depending on how we set things up. For example, here are two ways of defining the same thing:

   If $f : \mathbb{R}^2 \to \mathbb{R}$, and the curve C is defined using the parametric function $\mathbf{r}(t)$, then

   $$\int_C f(\mathbf{r}(t))\,ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\,dt$$

   Exercise: Evaluate $\int_C x^2 + xy\,ds$, where C is the upper half of the unit circle.

   SOLUTION: The purpose of this problem and the next was to get you back into your Calculus text to review the idea of line integral and the notation used. In this case,

   $$\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle \quad \Rightarrow \quad |\mathbf{r}'(t)| = \sqrt{(-\sin(t))^2 + (\cos^2(t))} = 1$$

   Therefore, once we translate to an integral in t, we did the work in a previous exercise:

   $$\int_C f(\mathbf{r}(t))\,ds = \int_0^{\pi/2} \cos^2(t) + \cos(t)\sin(t)\,dt = \frac{1}{2}t + \frac{1}{4}\sin(2t) + \frac{1}{2}\sin^2(t)\Big|_0^{\pi/2} = \frac{\pi}{4} + \frac{1}{2}$$

6. (*)**Review Question, Calc 3**: If $\mathbf{F}$ is a vector field on a smooth curve C defined by $\mathbf{r}(t)$, then the work done by $\mathbf{F}$ in moving a particle along the curve C is given by

   $$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

   Exercise: Find the work done using

   $$\mathbf{F} = \langle x, y, xy \rangle \qquad \mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle \qquad 0 \le t \le \pi$$

   SOLUTION TO BE POSTED.

7. From Section 1.2:

   (a) (*) 1(a, b) SOLUTION TO BE POSTED

   (b) Exercise 2 SOLUTION: The idea is to use the "exact" form (we called it an alternative form) of the conservation of heat energy on page 4. That is, the total thermal energy (that was approximated by $e(x,t)A\,\Delta x$), would be written as the integral:

   $$\int_x^{x+\Delta x} e(x,t)A\,dx$$

   so that the conservation of heat energy would look like the following ($Q = 0$):

   $$\frac{d}{dt}\int_x^{x+\Delta x} e(x,t)A\,dx = (\phi(x,t) - \phi(x+\Delta x, t))A$$

Divide by A and rewrite the middle term as:

$$\int_x^{x+\Delta x} \phi_x(x,t)\,dx$$

(This is actually kind of sloppy notation, since the variable of integration is the same as the bounds, but we'll live with it to emphasize the similarity between this derivation and that in the text).

Therefore, bring all of the integrals on the left side and we have:

$$\int_x^{x+\Delta x} e_t(x,t) + \phi_x(x,t)\,dx = 0$$

Which is true for an arbitrary Δx. Therefore, as long as the integrand is continuous, we have:

$$e_t(x,t) = -\phi_x(x,t)$$

which we had before. We'll go ahead and use the relationship between energy and temperature (c is specific heat):

$$e(x,t) = c\,\rho\,u(x,t)$$

And Fourier's Law:

$$\phi = -K_0 u_x$$

to get the heat equation as before.

The second part of Exercise 2 is exactly the same, except the bounds of the integral are an arbitrary a, b instead of $x, x + \Delta x$.

(c) Exercise 8 SOLUTION:

We had this in class, and also see the note before Equation 1.2.4 on p. 4. If c is the specific heat (which could be a function of x), and ρ is the density of the rod (which could be a function of x), then the total energy in the rod is given by:

$$\int_0^L e(x,t)A\,dx = \int_0^L c\,\rho\,u(x,t)\,A\,dx$$