Computer Lab, Math 367: Vibrating Strings[1]

A string which undergoes small transverse[2] vibrations satisfies the one dimensional wave equation:

$$u_{tt} = c^2 u_{xx} \qquad 0 < x < L$$

where L is the length of the string. If T is the tension (assumed constant) and ρ is the density (assumed constant), then $c = \sqrt{T/\rho}$ is the velocity of the wave, or the propagation speed.

If we assume the following boundary and initial conditions:

$$\begin{array}{lll} \text{BCs} & u(0,t) = 0 & u(L,t) = 0 \\ \text{ICs} & u(x,0) = f(x) & u_t(x,0) = g(x) \end{array}$$

then the method of eigenfunctions will use:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad X_n = \sin\left(\frac{n\pi}{L}\,x\right)$$

And leads us to solve the time ODE, which gives:

$$T_n(t) = A_n \cos\left(\frac{n\pi c}{L}\,t\right) + B_n \sin\left(\frac{n\pi c}{L}\,t\right)$$

That leads us to the overall solution:

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(A_n \cos\left(\frac{n\pi c}{L}\,t\right) + B_n \sin\left(\frac{n\pi c}{L}\,t\right)\right)$$

where

$$u(x,0) = \sum A_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \quad \Rightarrow \quad A_n = \frac{2}{L}\int_0^L f(x)\,\sin\left(\frac{n\pi}{L}x\right)\,dx$$

and

$$u_t(x,0) = \sum B_n \frac{n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right) = g(x) \quad \Rightarrow \quad B_n = \frac{L}{n\pi c}\,\frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)\,dx$$

Vibrations of a Plucked String

Suppose a string is attached at $x = 0$ and $x = L$, and assume $c = 1$. Then if we pluck a string and release it, then the initial velocity is $g(x) = 0$ and the initial position might be well modeled by $f(x)$ below.

$$f(x) = \begin{cases} 2x/L & \text{if } 0 < x \le L/2 \\ 2(L-x)/L & \text{if } L/2 < x < L \end{cases} \tag{1}$$

[1]Based on computer labs from Northeastern U. by R. McOwen.

[2]Perpendicular to the x axis

The graph of this function looks like an inverted "V": The graph goes in a line from the origin $(0,0)$ to $(L/2,1)$, then back down to the $x-$axis, $(L,0)$.

We could write f piecewise in Maple (the command is `piecewise`), the function only comes in when we compute the coefficients A_n. If we take $L=1$ to simplify the notation, then:

$$A_n = 2\left(\int_0^{1/2} 2x\sin(n\pi x)\,dx + \int_{1/2}^1 2(1-x)\sin(n\pi x)\,dx\right)$$

Furthermore, since $g(x)=0$, we'll have $B_n=0$. We can define the N^{th} partial sum of the Fourier series to the solution as the following (recall that $c=1$ and $L=1$):

$$S_N(x,t) = \sum_{n=1}^{N} A_n \cos(n\pi\, t)\sin(n\pi x)$$

It is not difficult to compute A_n (using integration by parts), but why not ask *Maple* to do it? We will enter:

```
F1:=2*x*sin(n*Pi*x); F2:=2*(1-x)*sin(n*Pi*x);
A:=n->2*(int(F1,x=0..1/2)+int(F2,x=1/2..1));
S:=(N,x,t)->sum(A(n)*cos(n*Pi*t)*sin(n*Pi*x),n=1..N);
```

To see the 5th partial sum which approximates the solution $u(x,t)$, we would type $S(5,x,t)$. We can plot this as well:

```
S(5,x,t);
plot3d(S(5,x,t),x=0..1,t=0..4,shading=zhue,axes=boxed);
```

We can also plot various "snapshots" of the approximate solution. For example, to see them at $t=0, t=0.2, t=0.8, t=1.2$, we would type:

```
plot({S(5,x,0), S(5,x,0.2),S(5, x,0.8), S(5, x,1.2)}, x = 0..1, axes = normal);
```

You can also animate it:

```
with(plots):  #This should really be at the top of the Maple worksheet
animate(plot, [S(5,x,t),x=0..1],t=0..4);
```

If you'd like to keep the animation as a separate file, Maple can export it to an animated GIF. Right-click on the plot, and choose `Export` then `GIF`, then choose a file folder and file name. You can watch the animation using any web browser or most image viewing programs.

You should see periodic wave motion- In this case, the period of $S(5,x,t)$ is the smallest value of t^* so that

$$S(5,x,t) = S(5,x,t+t^*)$$

See if you can graphically estimate the period, then verify it by looking at $S(5,x,0)$ and $S(5,x,t^*)$.

Exercises

1. Change the value of $L = 2$, and make the appropriate changes to the Maple code.

   (a) Obtain a 3d plot of the partial sum $S(5, x, t)$ for $0 < x < 2$ and $0 < t < 4$.

   (b) Animate the solution using the same values of x and t as the 3-d plot. Export the animation as a GIF file.

   (c) Is this motion periodic? If so, see if you can estimate the period, then verify it by looking at $S(5, x, t)$ and $S(5, x, t^*)$.

   (d) Save the Maple file and upload it to your CLEo dropbox.

2. Suppose that instead of plucking the string, we strike the string with a hammer. Practically speaking, that means that $f(x) = 0$ and now we'll model the velocity function g, on the interval $0 < x < 4$, by the function:

$$g(x) = \begin{cases} 1 & \text{for } 1 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

   The effect of these changes to our solution is to make $A_n = 0$, and

$$B_n = \frac{L}{n\pi c} \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{n\pi} \int_1^3 \sin\left(\frac{n\pi}{4} x\right) dx$$

   Now,

$$S_N(x, t) = \sum_{n=1}^{N} B_n \sin\left(\frac{n\pi t}{4}\right) \sin\left(\frac{n\pi x}{4}\right)$$

   Modify the previous Maple worksheet to give `S(5,x,t)` under the new conditions.

   (a) Give a 3d plot of $S_5(x, t)$ for $0 < x < 4$ and $0 < t < 4$.

   (b) Plot some slices of $S_5(x, t)$.

   (c) Is the motion periodic? Estimate and verify, as before.

   (d) If you hit the string with more velocity, would the period change?

   (e) Save your Maple file with solutions, and upload to CLEo.

3. Plot some of the normal modes of vibration for a string- That is, see if you can reproduce the 3d plots in Figure 4.4.1, p 141 (NOTE: Don't try to get it exactly the same; there are some scaling issues with the text figures, and the wave at $t = 0$ isn't clear). Are there any points in each plot that never move? (Yes, these are called *nodes*). See if you can identify them for the three plots you produced. You may assume $c = L = 1$. Upload the Maple file to CLEo.

4. Use the Maple file from our class website, `LeftRightWaves.mw`, which gives the solution to the wave equation using Exercise 4.4.7.

   Modify the file to solve the wave equation with $g(x) = 0$ and $f(x)$ given in Equation 1. Animate the solution, export to a GIF file, and upload it to CLEo (just the GIF file).

   As a side note, you might compare this exact solution with the approximate solution we found in the first exercise (and the period).