Notes about Exam 1, Math 367

The exam will be about 50 minutes in length, and no calculators (you really shouldn't need them). No notes or textbooks will be allowed.

Some formulas will be provided, if necessary, such as: Cosine/sine sum identities (for the trig integrals), and things like the polar form of Laplace's equation.

You should recall how to integrate $\int_0^L \sin^2(n\pi x/L)\,dx$, or just remember that it's $L/2$ (we'll go through these in detail in Chapter 3).

Sample Questions, Exam 1

1. Is it always true that if $\displaystyle\int_a^b f(x)\,dx = 0$, then $f(x) = 0$? What conditions did we need to make it true?

2. Explain the negative sign in Fourier's law.

3. Use the simplified heat equation, $u_t = ku_{xx}$, and suppose that our solution is of the form $u(x,t) = p_1(t) + p_2(x)$ where p_1 is a polynomial in t, and p_2 is a polynomial in x. Find p_1, p_2.

4. Show that $u(x,y) = \ln(x^2 + y^2)$ solves Laplace's equation, as long as $(x,y) \neq (0,0)$.

5. Heat flow in a metal rod with an internal heat source is modeled by the following PDE:

$$\begin{array}{lll} u_t = ku_{xx} + 1 & & 0 < x < L, t > 0 \\ u(0,t) = 0 & u(L,t) = 1 & t > 0 \end{array}$$

   Find the equilibrium solution, if it exists.

6. Suppose that the set of functions $\{\phi_1(x), \phi_2(x), \cdots, \phi_n(x)\}$ is an orthogonal set on the interval $[a,b]$, and that

$$f(x) = c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x)$$

   Find a formula (show your work!) for the constants c_i.

7. Suppose that $u(x,y)$ is the temperature of a solid region R in the plane, and $k = u(x,y)$ is a level curve for which the temperature is k degrees, and the point (a,b) is on the level curve. Let $\vec{\phi}(x,y)$ be the heat flux.

   (a) Is each quantity a scalar (label with S) or a vector (label with V)?

   - $u(x,y)$
   - $\vec{\phi}(x,y)$
   - $\nabla u(x,y)$

- $\nabla^2 u(x,y)$

(b) In what direction (in words) does $\nabla u(a,b)$ point?

(c) If the heat flux is $\vec{\phi}(x,y)$, how was Fourier's law interpreted? (That is, what is the result of Fourier's law in 2-d?)

8. Consider the interval $[0,\pi]$, and suppose we want to write $f(x)$ using an appropriate sum of sine functions,

$$f(x) = \sum_{n=1}^{\infty} B_n \sin(???)$$

Find an expression to replace the question marks, then find the expression for the n^{th} coefficient. Your work should show that you understand how we get the formulas.

9. Suppose that both ends of the rod are insulated, and there is no heat source. Show that the total thermal energy in the rod is constant, and find the equilibrium solution if $u(x,0) = f(x)$.

10. Using the conservation of heat energy, we said that, for a rod going from $x = a$ to $x = b$, the rate of change of the total energy in the rod is given by:

$$\frac{d}{dt}\int_a^b e(x,t)\,dx = \phi(a,t) - \phi(b,t) + \int_a^b Q(x,t)\,dx$$

(a) Define the functions e, ϕ, Q in that expression (what are they?):

(b) Justify the middle expression: $\phi(a,t) - \phi(b,t)$. What does it represent, and why are we subtracting?

(c) Show that the equation given can be written as the following:

$$\int_a^b e(x,t) + \phi_x(x,t) + Q(x,t)\,dx = 0$$

(d) Is it always true that if $\int_a^b f(x)\,dx = 0$, then $f(x) = 0$?

(e) Starting with equation (10c), show the following, assuming K_0 is constant. Be explicit about any relationships you're using.

$$c\rho u_t(x,t) = K_0 u_{xx} + Q$$

11. Suppose we have a rod of length L which has its sides insulated, and whose right end (at $x = L$) is not. If $u(x,t)$ is the temperature of the rod at position x and time t, show how Newton's Law of Cooling is used to construct the boundary condition if there is a constant environmental temperature of 5 degrees Celsius and the constant you use is positive. (Hint: You should first state what Newton's Law of Cooling says in words).

12. Using the equation given in 10, suppose that $a = 0$ and $b = L$, and there are no heat sources or sinks in the rod. Show that, if the ends of the rod are insulated, then we can conclude:

$$\int_0^L c\rho u(x,t)\,dx = C$$

where C is an arbitrary constant (which means the total energy in the rod is constant in time).

13. Continuing with the previous problem, if $u(x,0) = f(x)$ and there is a constant equilibrium,

$$\lim_{t\to\infty} u(x,t) = C_2$$

Show that C_2 is the average value of f:

$$C_2 = \frac{1}{L}\int_0^L f(x)\,dx$$

(Hint: We can make appropriate substitutions in 12 in at least two ways.)

14. It can be shown that, under certain circumstances, if our rod is not laterally insulated, then the heat equation changes to:

$$u_t = ku_{xx} - \beta u$$

Show that by using the change of variables:

$$u(x,t) = \mathrm{e}^{-\beta t}w(x,t)$$

that the PDE for w is: $w_t = kw_{xx}$.

15. Given the following PDE:

$$\begin{array}{ll} \text{PDE} & u_t = u_{xx} + x \\ \text{BCs} & u_x(0,t) = \beta, \quad u_x(1,t) = 3 \\ \text{ICs} & u(x,0) = x \quad 0 < x < 1 \end{array}$$

(a) Calculate the total thermal energy in the rod as a function of time.

(b) Determine a value of β for which an equilibrium solution exists.

(c) Find the equilibrium solution.

16. Given $u_t = u_{xx}$ with $u(0,t) = T$ and $u(L,t) + u_x(L,t) = 0$, find the equilibrium solution (if one exists).

17. Solve Laplace's equation outside a circular disk $r \geq a$ subject to the given boundary condition: $u(a,\theta) = \ln(2) + 4\cos(3\theta)$. Some notes:

- The polar form of Laplace's equation is:

$$\frac{1}{r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta} = 0$$

- You may also assume periodic boundary conditions:
  $u(r,-\pi) = u(r,\pi)$ and $u_\theta(r,-\pi) = u_\theta(r,\pi)$

18. Solve Laplace's equation inside a 60° degree wedge, subject to the boundary condition $u_\theta(r,0) = 0$, $u_\theta(r,\pi/3) = 0$ and $u(a,\theta) = f(\theta)$. The polar form of Laplace's equation was given in the last problem.

19. What is the relationship between Laplace's equation and the heat equation (if any)?

20. Consider: $\nabla^2 u = 0$ over the rectangle $0 \le x \le L$, $0 \le y \le H$.

    (a) Explain how we break up the general solution to Laplace's equation over a rectangle into 4 "easier" problems.

    (b) Suppose that the appropriate boundary functions are $f_1(x), f_2(x), g_1(y)$ and $g_2(y)$, and that one of the solutions is:

    $$u_?(x,y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi}{L}(y-H)\right)\sin\left(\frac{n\pi}{L}x\right)$$

    where we've lost track of which solution this is... Which solution should it be, and give the appropriate formula for B_n.

    (c) Find the other three functions, also with appropriate formulas for their coefficients.

21. Solve Laplace's equation over the rectangle $0 \le x \le 1$, $0 \le y \le 1$, if the boundary conditions are:

$$u(0,y) = 0, \quad u(1,y) = 0, \quad u(x,0) - u_y(x,0) = 0 \quad u(x,1) = f(x)$$

22. Solve the eigenvalue problem $\frac{d^2\phi}{dx^2} = -\lambda\phi$, with $\phi(0) = \phi(2\pi)$ and $\phi'(0) = \phi'(2\pi)$. Be sure to consider three cases.

23. Give the general solution to the Euler equation:

    (a) $x^2y'' + 4xy' + 2y = 0$

    (b) $x^2y'' - 3xy' + 4y = 0$

24. For each PDE or boundary condition below, state whether or not it is LINEAR, and whether or not it is HOMOGENEOUS.

    (a) $u_x(0,t) = -H(u(0,t) - 30)$

(b) $u_t(x,t) = u_x(x,t)u(x,t)$

(c) $u(0,t) + u_x(0,t) = 0$

25. For each PDE, try using separation of variables to transform the equation into two ODEs (if possible). Do not solve the ODEs:

    (a) $xu_{xx} + u_t = 0$

    (b) $u_{xx} + (x+y)u_{yy} = 0$