Solutions to Exam 1 Review:

1. Is it always true that if $\int_a^b f(x)\,dx = 0$, then $f(x) = 0$? What conditions did we need to make it true?

   SOLUTION: In general, the statement is not true (think of sine or cosine from 0 to 2π, for example). However, if the integral is zero for arbitrary a, b, and f is continuous, then the statement is true- Mainly because if f is continuous and $f(x^*) > 0$ for some x^*, then there must be a small interval about x^* on which $f > 0$ (so the integral on that interval would not be zero).

2. Explain the negative sign in Fourier's law.

   SOLUTION: The way we defined flux was to say that it is the flow of heat **to the right**. Therefore, if u_x is positive at some point x, then the temperature is increasing as we move from left to right, which means that heat flux is moving in the opposite direction.

3. Use the simplified heat equation, $u_t = ku_{xx}$, and suppose that our solution is of the form $u(x,t) = p_1(t) + p_2(x)$ where p_1 is a polynomial in t, and p_2 is a polynomial in x. Find p_1, p_2.

   SOLUTION: Given our ansatz:

$$u(x,t) = p_1(t) + p_2(x) \quad \Rightarrow \quad u_t = ku_{xx} \quad \Rightarrow \quad \frac{dp_1}{dt} = k\frac{d^2p_2}{dx^2}$$

   The only way a function of t can be equal to a function of x is if they are both equal to the same constant. Let that constant be λ. Then

$$p_1'(t) = \lambda \quad \Rightarrow \quad p_1(t) = kt + C_1, \text{ for arbitrary scalars } C_1$$

$$kp_2''(x) = \lambda \quad \Rightarrow \quad p_2(x) = \frac{\lambda}{2k}x^2 + C_2x + C_3, \text{ for arbitrary } C_2, C_3$$

   If we put the solutions together to form $u(x,t)$, we get

$$u(x,t) = \lambda t + \frac{\lambda}{2k}x^2 + C_2x + C_4$$

4. Show that $u(x,y) = \ln(x^2 + y^2)$ solves Laplace's equation, as long as $(x,y) \neq (0,0)$.

   SOLUTION: We just substitute u into Laplace's equation to see if we get a true statement. We'll need the second derivatives- Notice that this problem is symmetric in x, y, so once we get the solution for x, it will be easy to get for y:

$$u = \ln(x^2 + y^2) \quad \Rightarrow \quad u_x = \frac{2x}{x^2 + y^2} \quad \Rightarrow \quad u_{xx} = \frac{2(x^2 + y^2) - 2x(2x)}{(x^2 + y^2)^2} = \frac{-2x^2 + 2y^2}{(x^2 + y^2)^2}$$

By symmetry, we also have:

$$u_{yy} = \frac{-2y^2 + 2x^2}{(x^2+y^2)^2}$$

And we see that $u_{xx} + u_{yy} = 0$.

5. Heat flow in a metal rod with an internal heat source is modeled by the following PDE:

$$\begin{array}{ll} u_t = ku_{xx} + 1 & 0 < x < L, t > 0 \\ u(0,t) = 0 \quad u(L,t) = 1 & t > 0 \end{array}$$

   Find the equilibrium solution, if it exists.

   SOLUTION: Let's see if the long term solution exists- If so, we must have:

$$0 = ku''(x) + 1 \quad \Rightarrow \quad u''(x) = -\frac{1}{k} \quad \Rightarrow \quad u(x) = -\frac{1}{2k}x^2 + C_1 x + C_2$$

   Matching initial conditions,

$$u(0) = 0 \quad \Rightarrow \quad C_2 = 0$$

$$u(L) = 1 \quad \Rightarrow \quad -\frac{L^2}{2k} + C_1 L = 1 \quad \Rightarrow \quad C_1 = \frac{1}{L}\left(1 + \frac{L^2}{2k}\right)$$

6. Suppose that the set of functions $\{\phi_1(x), \phi_2(x), \cdots, \phi_n(x)\}$ is an orthogonal set on the interval $[a, b]$, and that

$$f(x) = c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x)$$

   Find a formula (show your work!) for the constants c_i.

   SOLUTION: If we want c_i, then multiply both sides of the equation by $\phi_i(x)$, then integrate both sides from a to b:

$$\int_a^b f(x)\phi_i(x) = c_1 \int_a^b \phi_1(x)\phi_i(x)\,dx + \cdots + c_i \int_a^b \phi_i^2(x)\,dx + \cdots + c_n \int_a^b \phi_n(x)\phi_i(x)\,dx$$

   Every integral on the RHS is 0 by orthogonality (except for the one with c_i), so this simplifies to:

$$\int_a^b f(x)\phi_i(x)\,dx = c_i \int_a^b \phi_i^2(x)\,dx \quad \Rightarrow \quad c_i = \frac{\int_a^b f(x)\phi_i(x)\,dx}{\int_a^b \phi_i^2(x)\,dx}$$

7. Suppose that $u(x, y)$ is the temperature of a solid region R in the plane, and $k = u(x, y)$ is a level curve for which the temperature is k degrees, and the point (a, b) is on the level curve. Let $\vec{\phi}(x, y)$ be the heat flux.

(a) Is each quantity a scalar (label with S) or a vector (label with V)?

- $u(x, y)$ Scalar (S)
- $\vec{\phi}(x, y)$ Vector (V)
- $\nabla u(x, y)$ Vector (V), $\langle u_x, u_y \rangle$
- $\nabla^2 u(x, y)$ Scalar (S), $u_{xx} + u_{yy}$

(b) In what direction (in words) does $\nabla u(a, b)$ point?

SOLUTION: In the direction of maximum *increase* of temperature.

Extra note: This comes from the directional derivative computation. That is, the rate of change of $u(x, y)$ in the (unit) direction $\mathbf{v}$ is given by:

$$D_v u = \nabla u \cdot \mathbf{v} = |\nabla u| \cos(\theta)$$

where θ is the angle between ∇u and $\mathbf{v}$. Therefore, this quantity is maximum when $\theta = 0$.

(c) If the heat flux is $\vec{\phi}(x, y)$, how was Fourier's law interpreted? (That is, what is the result of Fourier's law in 2-d?)

SOLUTON: $\phi = -K_0 \nabla u$

Extra note: This is because ∇u points in the direction of maximum *increase* in temperature, so the flux will be in the other (opposite) direction.

8. Suppose $f(x) = 2x$, and we consider the interval $[0, \pi]$. Suppose we want to write $f(x)$ using an appropriate sum of sine functions,

$$f(x) = \sum_{n=1}^{\infty} B_n \sin(???)$$

Find an expression to replace the question marks, then find the expression for the n^{th} coefficient. Your work should show that you understand how we get the formulas.

SOLUTION: The sine term should be $(n\pi/L)x$, and L in this case is π, so the functions are: $B_n \sin(nx)$.

That means:

$$f(x) = B_1 \sin(x) + B_2 \sin(2x) + B_3 \sin(3x) + \cdots$$

To find the m^{th} coefficient, multiply both sides by $\sin(mx)$ and integrate from 0 to π:

$$\int_0^{\pi} f(x) \sin(mx)\, dx = B_1 \int_0^{\pi} \sin(x) \sin(mx)\, dx + \cdots + B_m \int_0^{\pi} \sin^2(mx)\, dx + \cdots$$

Using the orthogonality of the sines, and

$$\int_0^{\pi} \sin^2(mx)\, dx = \frac{\pi}{2}$$

the sum above simplifies to:

$$\int_0^{\pi} f(x)\sin(mx)\,dx = 0 + 0 + \cdots + B_m\frac{\pi}{2} + 0 + \cdots.$$

And solving for B_m leads us to the formula:

$$B_m = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(mx)\,dx$$

9. Suppose that both ends of the rod are insulated, and there is no heat source. Show that the total thermal energy in the rod is constant, and find the equilibrium solution if $u(x,0) = f(x)$.

SOLUTION: From what is given, we have the following PDE:

$$\begin{array}{ll} \text{PDE} & u_t = k u_{xx} \\ \text{BCs} & u_x(0,t) = u_x(L,t) = 0 \\ \text{ICs} & u(x,0) = f(x) \end{array}$$

Integrating both sides of the PDE with respect to x, we see that:

$$\frac{d}{dt}\int_0^L u(x,t)\,dx = k\int_0^L u_{xx}(x,t)\,dx = k(u_x(L,t) - u_x(0,t)) = k(0-0) = 0$$

Therefore, the energy in the rod, $\displaystyle\int_0^L c\rho u(x,t)A\,dx$, is constant.

The equilibrium solution is then found by setting $u_t = 0$, from which:

$$u_{xx} = 0 \quad \Rightarrow \quad u(x) = C_1 x + C_2$$

The derivative is C_1, so set it to zero, and we have: $u(x) = C_2$.

Now, to find the value of C_2, we note that $\int_0^L u(x,t)\,dx$ is constant, and so it can be computed at any time- In particular, at time 0, then as $t \to \infty$:

$$\int_0^L u(x,t)\,dx = \int_0^L u(x,0)\,dx = \int_0^L f(x)\,dx$$

And

$$\int_0^L u(x,t)\,dx = \int_0^L u(x)\,dx = \int_0^L c_2\,dx = C_2 L$$

therefore,

$$C_2 = \frac{1}{L}\int_0^L f(x)\,dx$$

10. Using the conservation of heat energy, we said that, for a rod going from $x = a$ to $x = b$, the rate of change of the total energy in the rod is given by:

$$\frac{d}{dt}\int_a^b e(x,t)\,dx = \phi(a,t) - \phi(b,t) + \int_a^b Q(x,t)\,dx$$

(a) Define the functions e, ϕ, Q in that expression (what are they?):

SOLUTION: e is the energy density per unit area, ϕ is the flux (to the right) per unit area per unit time, and $Q(x,t)$ is heat energy generated (per unit volume) per unit time.

(b) Justify the middle expression: $\phi(a,t) - \phi(b,t)$. What does it represent, and why are we subtracting?

SOLUTION: The flux is measuring the amount of heat energy flowing to the right. Therefore, for $a < b$, the quantity $\phi(a,t)$ is the amount of energy flowing into the rod from the left side and $-\phi(b,t)$ is the amount of energy flowing into the rod from the right side (so we subtract to denote the leftward flow from the right side).

(c) Show that the equation given can be written as the following:

$$\int_a^b e(x,t) + \phi_x(x,t) + Q(x,t)\,dx = 0$$

SOLUTION: Using the Fundamental Theorem of Calculus,

$$-\int_a^b \phi_x(x,t)\,dx = \phi(a,t) - \phi(b,t)$$

Then bring all of the integrals over to the left side of the equation and collect them.

(d) Is it always true that if $\displaystyle\int_a^b f(x)\,dx = 0$, then $f(x) = 0$?

SOLUTUION: No- For example, $\int_0^{2\pi} \sin(x)\,dx$. However, it is true with two changes- We want f to be continuous, and the integral should be zero for *any* choice of $a < b$. Then the statement is true.

(e) Starting with equation (10c), show the following, assuming K_0 is constant. Be explicit about any relationships you're using.

$$c\rho u_t(x,t) = K_0 u_{xx} + Q$$

SOLUTION: There are two key relationships being used:

- From the definition of energy density, $e(x,t) = c\rho u(x,t)$
- From Fourier's law: $\phi = -K_0 u_x$.

Then make the substitutions in Equation 10c.

11. Suppose we have a rod of length L which has its sides insulated, and whose right end (at $x = L$) is not. If $u(x, t)$ is the temperature of the rod at position x and time t, show how Newton's Law of Cooling is used to construct the boundary condition if there is a constant environmental temperature of 5 degrees Celsius and the constant you use is positive. (Hint: You should first state what Newton's Law of Cooling says in words).

    SOLUTION: Newton's Law of Cooling states that the rate of change of temperature of a body is proportional to the difference between the temperature of the body and the environmental temperature.

    To translate that for a boundary condition for the right side of a rod, $x = L$, would imply the following (assuming a constant environmental temp of 5 degrees). We might start with something like:

    $$u_x(L, t) = H(u(L, t) - 5)$$

    where H is the constant of proportionality. We should check to see if H is positive or negative; we should make the equation true for a positive parameter. In this case, we want to put a negative sign in front of the H- For example, if the end of the rod is hot, then the temperature ought to be decreasing. Therefore, our final answer is:

    $$u_x(L, t) = -H(u(L, t) - 5)$$

    NOTE: The textbook keeps K_0 with u_x, but it is OK if you divide it out to get $H = h/K_0$.

12. Using the equation given in 10, suppose that $a = 0$ and $b = L$, and there are no heat sources or sinks in the rod. Show that, if the ends of the rod are insulated, then we can conclude:

    $$\int_0^L c\rho u(x, t)\, dx = C$$

    where C is an arbitrary constant (which means the total energy in the rod is constant in time).

    SOLUTION: This is basically the same as Problem 3. Notice that in the next problem, we continue to determine the value of the constant.

13. Continuing with the previous problem, if $u(x, 0) = f(x)$ and there is a constant equilibrium,

    $$\lim_{t\to\infty} u(x, t) = C_2$$

    Show that C_2 is the average value of f:

    $$C_2 = \frac{1}{L}\int_0^L f(x)\, dx$$

    (Hint: We can make appropriate substitutions in 12 in at least two ways.)

    SOLUTION: See Problem 3.

14. It can be shown that, under certain circumstances, if our rod is not laterally insulated, then the heat equation changes to:

$$u_t = k u_{xx} - \beta u$$

Show that by using the change of variables:

$$u(x,t) = \mathrm{e}^{-\beta t} w(x,t)$$

that the PDE for w is: $w_t = k w_{xx}$.

SOLUTION: We'll substitute expressions for u_t, u_{xx} in the given PDE and see what we get. Remember to use the product rule:

$$u_t = -\beta \mathrm{e}^{-\beta t} w(x,t) + \mathrm{e}^{-\beta t} w_t(x,t)$$

Similarly,

$$u_{xx} = \mathrm{e}^{-\beta t} w_{xx}(x,t)$$

Therefore, substituting these into $u_t = k u_{xx} - \beta u$, we get:

$$-\beta \mathrm{e}^{-\beta t} w(x,t) + \mathrm{e}^{-\beta t} w_t(x,t) = k(\mathrm{e}^{-\beta t} w_{xx}(x,t)) - \beta \cdot \mathrm{e}^{-\beta t} w(x,t)$$

Simplifying, we get: $w_t = k w_{xx}$.

15. Given the following PDE:

$$\begin{array}{ll} \text{PDE} & u_t = u_{xx} + x \\ \text{BCs} & u_x(0,t) = \beta, \quad u_x(1,t) = 3 \\ \text{ICs} & u(x,0) = x \quad 0 < x < 1 \end{array}$$

(a) Calculate the total thermal energy in the rod as a function of time.

SOLUTION: Instead of calculating the energy directly, we have an expression for the rate of change of energy in time- Just integrate the PDE by x, and:

$$\frac{d}{dt}\int_0^1 u(x,t)\,dx = \int_0^1 u_{xx} + x\,dx = u_x(1,t) - u_x(0,t) + \frac{1}{2} = 3 - \beta + \frac{1}{2}$$

Therefore,

$$\int_0^1 u(x,t)\,dx = \left(\frac{7}{2} - \beta\right) t + C$$

We can solve for C by using the initial temperature profile:

$$\int_0^1 u(x,0)\,dx = \int_0^1 x\,dx = 0 + C \quad \Rightarrow \quad C = \frac{1}{2}$$

therefore,

$$\int_0^1 u(x,t)\,dx = \left(\frac{7}{2} - \beta\right) t + \frac{1}{2}$$

Finally, we should multiply both sides by $c\rho A$.

(b) Determine a value of β for which an equilibrium solution exists.

SOLUTION: From the previous question, the change in time is zero if $\beta = 7/2$.

(c) Find the equilibrium solution.

SOLUTION: Given $\beta = 7/2$, the equilibrium solution solves the ODE:

$$0 = u_{xx} + x \quad u'(0) = \frac{7}{2} \quad u'(1) = 3$$

Continuing,

$$u_x = -\frac{1}{2}x^2 + C_1 \quad \Rightarrow \quad u(x) = -\frac{1}{6}x^3 + C_1 x + C_2$$

We see that $u'(0) = \frac{7}{2}$ implies that $C_2 = \frac{7}{2}$ and

$$u'(1) = -\frac{1}{2} + \frac{7}{2} = 3$$

so that both boundary conditions are met. The constant C_2 can be determined by computing the total energy, which does not change with the given value of β. Therefore, the following totals should all be the same:

$$\int_0^1 u(x,t)\,dx = \int_0^1 u(x,0)\,dx = \int_0^1 u(x)\,dx$$

Taking the last two,

$$\int_0^1 x\,dx = \int_0^1 -\frac{1}{6}x^3 + \frac{7}{2}x + C_2\,dx \quad \Rightarrow \quad C_2 = -\frac{29}{24}$$

Notice that with this constant, the quantity $\int_0^1 u(x,t)\,dx$ stays at 1/2 for all $t \geq 0$.

16. Given $u_t = u_{xx}$ with $u(0,t) = T$ and $u(L,t) + u_x(L,t) = 0$, find the equilibrium solution (if one exists).

SOLUTION: Let's go ahead and try to find a solution. Make $u_t = 0$, and

$$u'' = 0 \quad \text{with} \quad u(0) = T \text{ and } u(L) + u'(L) = 0$$

We have:

$$u(x) = C_1 x + C_2$$

with $u(0) = T$, we have $C_2 = T$ and $u(x) = C_1 x + T$. Now considering the second boundary value:

$$u(L) + u'(L) = 0 \quad \Rightarrow (C_1 L + T) + C_1 = 0 \quad \Rightarrow \quad C_1 = \frac{-T}{1+L}$$

Therefore, the equilibrium solution is:

$$u(x) = \frac{-T}{1+L}x + T$$

Notice that $u_t = u_{xx} = 0$ and that $u(x)$ satisfies all boundary conditions.

17. Solve Laplace's equation outside a circular disk $r \geq a$ subject to the given boundary condition: $u(a, \theta) = \ln(2) + 4\cos(3\theta)$. Some notes:

   - The polar form of Laplace's equation is:
   $$\frac{1}{r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta} = 0$$
   - You may also assume periodic boundary conditions:
   $u(r, -\pi) = u(r, \pi)$ and $u_\theta(r, -\pi) = u_\theta(r, \pi)$

   SOLUTION (NOTE: This is exercise 2.5.3)

   Set up the separation of variables and the boundary conditions:
   $$u(r, \theta) = R(r)T(\theta)$$
   with $R(a) \neq 0 \quad T(\pi) = T(-\pi)$ and $T'(\pi) = T'(-\pi)$. Substitute the product into the PDE:
   $$\frac{1}{r}\left(rR'T\right)' + \frac{1}{r^2}RT'' = 0$$
   Multiply both sides by r^2/RT (note that T is constant in $(R'T)'$ because this is the derivative with respect to r) to get:
   $$\frac{r(rR')'}{R} + \frac{T''}{T} = 0 \quad \Rightarrow \quad \frac{r(rR')'}{R} = -\frac{T''}{T} = \lambda$$
   Now we have the three cases to consider:

   - Case 1: $\lambda = 0$
   $$\begin{aligned} r(rR')' &= 0 \\ rR' &= c \\ R' &= c/r \\ R &= c\ln(r) + c_2 \end{aligned} \qquad \Bigg| \qquad \begin{aligned} T'' &= 0 \\ T(\theta) &= C_1\theta + C_2 \end{aligned}$$
   Note: This form of the DE for R is easier than what we ended up doing in class; that was still accurate, but much longer.

     Applying the initial conditions, $T(\pi) = T(-\pi)$ will imply that $C_1 = 0$, and that gives $T'(\theta) = 0$. Therefore, $T(\theta) = C_2$ would be one possible solution. We also typically want our solution to be bounded. Since the possible values of r are $r \geq a$, the term with $\ln(r)$ will become unbounded as $r \to \infty$, so we set $c = 0$ in the DE for R.

     CONCLUSION: For $\lambda = 0$, R_0T_0 is constant.

- Now consider $\lambda < 0$. Normally, we might have to look at both R and T, but I think that T will be trivial in this case, which we check:

$$T'' + \lambda T = 0 \quad \Rightarrow \quad T(\theta) = C_1 \cosh(\sqrt{-\lambda}\theta) + C_2 \sinh(\sqrt{-\lambda}\theta)$$

With the given boundary conditions and using the fact that the hyperbolic cosine is even and the hyperbolic sine is odd, we get

$$T(-\pi) = T(\pi) \quad \Rightarrow \quad C_2 \sinh(\sqrt{-\lambda}\pi) = 0$$

which implies that $C_2 = 0$. Similarly, $T'(-\pi) = T'(\pi)$ forces $C_1 = 0$, so the only solution is $T(\theta) = 0$.

- For the final case, $\lambda > 0$. In this situation, we solve both ODEs:

$$\begin{aligned} r^2 R'' + rR' - \lambda R &= 0 \\ r(r-1) + r - \lambda &= 0 \\ r &= \pm\sqrt{\lambda} \\ R(r) &= C_1 r^{\sqrt{\lambda}} + C_2 r^{-\sqrt{\lambda}} \end{aligned} \qquad \Bigg| \qquad \begin{aligned} T'' + \lambda T &= 0 \\ T(\theta) &= C_1 \cos(\sqrt{\lambda}\theta) + C_2 \sin(\sqrt{\lambda}\theta) \end{aligned}$$

Applying the first initial condition $T(\pi) = T(-\pi)$ implies that

$$C_2 \sin(\sqrt{\lambda}\pi) = 0$$

Unlike the previous case, this equation has nontrivial solutions when

$$\sqrt{\lambda}\pi = n\pi \text{ for } n = 1, 2, 3, \cdots \quad \Rightarrow \quad \lambda = n^2 \text{ for } n = 1, 2, 3, \cdots$$

And in fact, we'll get the same equation if we solve $T'(-\pi) = T'(\pi)$.

Looking at our solutions for $R(r)$, if we want our solutions to remain bounded, we take $C_1 = 0$ and keep the other part of the solution. In summary we have:

$$R_n T_n = A_n r^{-n} \cos(n\theta) + B_n r^{-n} \sin(n\theta) \text{ for } n = 1, 2, 3, \cdots$$

Overall, we have the general solution by summing over all the eigenfunctions:

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} A_n r^{-n} \cos(n\theta) + \sum_{n=1}^{\infty} B_n r^{-n} \sin(n\theta)$$

Finally, we consider the initial condition:

$$u(a, \theta) = \ln(2) + 4\cos(3\theta) = A_0 + \sum_{n=1}^{\infty} A_n a^{-n} \cos(n\theta) + \sum_{n=1}^{\infty} B_n a^{-n} \sin(n\theta)$$

Equating coefficients, we see that:

$$\ln(2) = A_0 \qquad 4 = A_3 a^{-3} \quad \Rightarrow \quad A_3 = 4a^3$$

and every other coefficient is 0.

The solution to our PDE with BCs and IC is:

$$u(r,\theta) = \ln(2) + 4a^3 r^{-3}\cos(3\theta)$$

and this is valid for $0 \leq \theta \leq 2\pi$ and $r \geq a$.

Extra NOTE: For more practice, try solving Laplace's equation on an annulus, $a \leq r \leq b$ with $u_r(a,\theta) = 0$, $u(b,\theta) = g(\theta)$ and periodic boundary conditions. The solution will be posted as extra practice.

18. Solve Laplace's equation inside a 60° degree wedge, subject to the boundary condition $u_\theta(r,0) = 0$, $u_\theta(r,\pi/3) = 0$ and $u(a,\theta) = f(\theta)$. The polar form of Laplace's equation was given in the last problem.

    SOLUTION: *Side Note: This is exercise 2.5.7(b)*

    This is very similar to the last problem, so we'll pick it up where we began testing cases for λ. But first, note the new boundary conditions:

    $$T'(0) = 0 \quad T'(\pi/3) = 0 \quad R(a) \neq 0$$

    As usual, we'll also assume the solution is bounded, $|u(r,\theta)| < \infty$.

    - Case 1: $\lambda = 0$. In this case,

      $$R(r) = C_1 \ln(r) + C_2 \qquad T(\theta) = C_3\theta + C_4$$

      Since the solution is bounded, we'll have to have $C_1 = 0$. The zero derivatives will force $C_3 = 0$, therefore, for $\lambda = 0$, we have a constant solution.

    - Case 2: $\lambda < 0$. In this case,

      $$T(\theta) = C_1 \cosh(\sqrt{-\lambda}\theta) + C_2 \sinh(\sqrt{-\lambda}\theta)$$

      so that

      $$T'(0) = 0 \quad \Rightarrow \quad C_2\sqrt{-\lambda} = 0 \quad \Rightarrow \quad C_2 = 0$$

      Similarly, $T'(\pi/3) = 0$ implies that

      $$C_1\sqrt{-\lambda}\sinh(\sqrt{-\lambda}\pi/3) = 0$$

      The hyperbolic sine is zero only at zero (and $\sqrt{-\lambda}\pi/3 \neq 0$), so that forces $C_1 = 0$. Therefore, we have only the trivial solution if $\lambda < 0$.

    - Case 3: $\lambda > 0$. I can almost copy and paste the previous part of the answer, with some key changes:

      $$T(\theta) = C_1 \cos(\sqrt{\lambda}\theta) + C_2 \sin(\sqrt{\lambda}\theta)$$

so that

$$T'(0) = 0 \quad \Rightarrow \quad C_2\sqrt{-\lambda} = 0 \quad \Rightarrow \quad C_2 = 0$$

Similarly, $T'(\pi/3) = 0$ implies that

$$C_1\sqrt{\lambda}\sin(\sqrt{\lambda}\pi/3) = 0$$

Now, this implies that:

$$\sqrt{\lambda}\frac{\pi}{3} = n\pi \text{ for } n = 1, 2, 3, \cdots \quad \Rightarrow \quad \lambda = 9n^2 \text{ for } n = 1, 2, 3, \cdots$$

Therefore,

$$T_n(\theta) = A_n\cos(3n\theta)$$

Going back to solve for $R(r)$, if the ansatz if $R = r^k$, then

$$r^2R'' + rR' - 9n^2R = 0 \quad \Rightarrow \quad k^2 = 9n^2 \quad \Rightarrow \quad k = \pm 3n$$

Therefore, $R_n(r) = C_1r^{3n} + C_2r^{-3n}$. Keeping our solution bounded, we'll only keep the first one, so that

$$R_nT_n = A_nr^{3n}\cos(3n\theta)$$

and the overall solution is:

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} A_nr^{3n}\cos(3n\theta)$$

where

$$f(\theta) = u(a, \theta) = A_0 + A_1a^3\cos(3\theta) + A_2a^6\cos(6\theta) + A_3a^9\cos(9\theta) + \cdots$$

To find A_0, just integrate with respect to θ, and each term on the right (except A_0) will be zero:

$$\int_0^{\pi/3} f(\theta)\, d\theta = A_0\int_0^{\pi/3} d\theta + 0 + 0 + \cdots \quad \Rightarrow \quad A_0 = \frac{3}{\pi}\int_0^{\pi/3} f(\theta)\, d\theta$$

To find the m^{th} coefficient, multiply both sides by $\cos(3m\theta)$ and integrate. The right side will mostly be zero due to the orthogonality of the cosines.

$$\int_0^{\pi/3} f(\theta)\cos(3m\theta)\, d\theta = 0 + 0 + \cdots + A_ma^{3m}\int_0^{\pi/3}\cos^2(3m\theta)\, d\theta + 0 + 0 + \cdots$$

$$\int_0^{\pi/3} f(\theta)\cos(3m\theta)\, d\theta = A_ma^{3m}\frac{\pi}{6} \quad \Rightarrow \quad A_m = \frac{6}{a^{3m}\pi}\int_0^{\pi/3} f(\theta)\cos(3m\theta)\, d\theta$$

19. What is the relationship between Laplace's equation and the heat equation (if any)?

    The solution to Laplace's equation gives the steady state solution to the associated heat equation.

20. Consider: $\nabla^2 u = 0$ over the rectangle $0 \le x \le L$, $0 \le y \le H$.

    (a) Explain how we break up the general solution to Laplace's equation over a rectangle into 4 "easier" problems.

    SOLUTION: The overall solution (with four non-zero boundary functions) is broken up into four problems, where each only has one non-zero boundary. The overall solution will then be the sum of the four "simpler" solutions.

    (b) Suppose that the appropriate boundary functions are $f_1(x), f_2(x), g_1(y)$ and $g_2(y)$, and that one of the solutions is:

$$u_?(x,y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi}{L}(y-H)\right) \sin\left(\frac{n\pi}{L}x\right)$$

    where we've lost track of which solution this is... Which solution should it be, and give the appropriate formula for B_n.

    SOLUTION: We see that, for the unknown function

$$u_?(0,y) = 0 \qquad u_?(L,y) = 0 \qquad u_?(x,H) = 0$$

    The only non-zero boundary is $u_?(x,0)$, which is the "bottom" function $f_1(x)$, so we called this $u_1(x,y)$. In that case, the coefficients are:

$$u_1(x,0) = f_1(x) = \sum_{n=1}^{\infty} \left[B_n \sinh\left(\frac{n\pi}{L}(-H)\right)\right] \sin\left(\frac{n\pi}{L}x\right)$$

    The quantity in the square bracket is our coefficient for $f_1(x)$:

$$B_n \sinh\left(\frac{n\pi}{L}(-H)\right) = \frac{2}{L}\int_0^L f_1(x) \sin\left(\frac{n\pi}{L}x\right)\, dx$$

    so then divide by the hyperbolic sine to solve for B_n.

    (c) Find the other three functions, also with appropriate formulas for their coefficients.

    SOLUTION: For the other functions,

    - $u_2(x,y)$ (has the top function non-zero), we take

$$u_2(x,y) = u_1(x, H-y)$$

and replace $f_1(x)$ by $f_2(x)$ so that $u_2(x,0) = u_1(x,H) = 0$ and $u_2(x,H) = u_1(x,0) = f_2(x)$:

$$u_2(x,y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi}{L}(H-y-H)\right)\sin\left(\frac{n\pi}{L}x\right) =$$

$$\sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi}{L}(-y)\right)\sin\left(\frac{n\pi}{L}x\right)$$

with

$$B_n = \frac{2}{L\sinh\left(\frac{n\pi}{L}(-H)\right)}\int_0^L f_2(x)\sin\left(\frac{n\pi}{L}x\right)\,dx$$

- For $u_3(x,y)$ (with the left function $g_1(y)$ the only non-zero boundary function), take u_1 and swap x, y (and swap f_1 for g_1, and H, L):

$$u_3(x,y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi}{H}(x-L)\right)\sin\left(\frac{n\pi}{H}y\right)$$

  with

$$B_n = \frac{2}{H\sinh\left(\frac{n\pi}{H}(-L)\right)}\int_0^H g_1(y)\sin\left(\frac{n\pi}{H}y\right)\,dy$$

- For $u_4(x,y)$ (with the right function $g_2(y)$ the only non-zero boundary function), take u_2 and do the x, y swap:

$$u_4(x,y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi}{H}(-x)\right)\sin\left(\frac{n\pi}{H}y\right)$$

  with

$$B_n = \frac{2}{H\sinh\left(\frac{n\pi}{H}(-L)\right)}\int_0^H g_2(y)\sin\left(\frac{n\pi}{H}y\right)\,dy$$

21. Solve Laplace's equation over the rectangle $0 \le x \le 1$, $0 \le y \le 1$, if the boundary conditions are:

$$u(0,y) = 0, \quad u(1,y) = 0, \quad u(x,0) - u_y(x,0) = 0 \quad u(x,1) = f(x)$$

SOLUTION: As usual, we'll take the ansatz $u = XY$, and we notice that the boundary conditions become:

$$X(0) = 0 \quad X(1) = 0 \quad Y(0) - Y'(0) = 0 \quad Y(1) \neq 0$$

Therefore, we expect that X will have the periodic solutions and Y will have the hyperbolic sine/cosine. Substitution into Laplace's equation and selecting λ:

$$X''Y + XY'' = 0 \quad \Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

This yields the ODEs:
$$X'' + \lambda X = 0 \qquad Y'' - \lambda Y = 0$$
and now we consider the different cases for λ:

- For $\lambda = 0$: $X(x) = C_1 x + C_2$. If $X(0) = 0$, then $C_2 = 0$. If $X(1) = 0$, then $C_1 = 0$. Therefore, in this case, X is only the trivial solution, $X = 0$.
- For $\lambda < 0$: In this case, Y would be periodic and
$$X(x) = C_1 \cosh(\sqrt{-\lambda}x) + C_2 \sinh(\sqrt{-\lambda}x)$$
If $X(0) = 0$, then $C_1 = 0$. If $X(1) = 0$, then
$$C_2 \sinh(\sqrt{-\lambda}) = 0 \quad \Rightarrow \quad C_2 = 0$$
so again, X is only the trivial solution.
- For $\lambda > 0$,
$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$
With the boundary condition $X(0) = 0$, $A = 0$, so with $X(1) = 0$, we have:
$$B\sin(\sqrt{\lambda}) = 0 \quad \Rightarrow \quad \sqrt{\lambda} = n\pi \text{ for } n = 1, 2, 3, \cdots$$
or we can write $\lambda = n^2\pi^2$ for $n = 1, 2, 3, \cdots$, and the solution for X is:
$$X_n = B_n \sin(n\pi x)$$
The general solution for Y is:
$$Y = C_1 \cosh(n\pi y) + C_2 \sinh(n\pi y)$$
We have $Y(1) \neq 0$, and $Y(0) - Y'(0) = 0$. Since $Y(0) = C_1$ and $Y'(0) = n\pi C_2$, then:
$$C_1 - n\pi C_2 \quad \Rightarrow \quad C_2 = \frac{C_1}{n\pi}$$
Therefore, the solution for Y_n is:
$$Y_n = C_1 \left(\cosh(n\pi y) + \frac{1}{n\pi}\sinh(n\pi y)\right)$$
We'll incorporate C_1 into the constant B_n to get the overall solution:
$$u(x, y) = \sum_{n=1}^{\infty} B_n \left(\cosh(n\pi y) + \frac{1}{n\pi}\sinh(n\pi y)\right)\sin(n\pi x)$$
Since $u(x, 1) = f(x)$, we can solve for the coefficients B_n in the usual manner:
$$B_n = \frac{2}{\cosh(n\pi) + \frac{1}{n\pi}\sinh(n\pi)} \int_0^1 f(x)\sin(n\pi x)\, dx$$

22. Solve the eigenvalue problem $\frac{d^2\phi}{dx^2} = -\lambda\phi$, with $\phi(0) = \phi(2\pi)$ and $\phi'(0) = \phi'(2\pi)$. Be sure to consider three cases.

SOLUTION: The three cases deal with λ.

- If $\lambda = 0$, then $\phi(x) = C_1 x + C_2$. If $\phi(0) = \phi(2\pi)$, then

$$C_2 = 2\pi C_1 + C_2 \quad \Rightarrow \quad C_1 = 0$$

  For the other condition, $\phi'(x) = 0$, so that is satisfied. Therefore, $\phi = C$ is one possible solution.

- If $\lambda < 0$, then the characteristic equation would be $r^2 + \lambda = 0$, or $r = \pm\sqrt{-\lambda}$, which would be two real numbers. Using the hyperbolic sine and cosine,

$$\phi(x) = C_1 \cosh(\sqrt{-\lambda}x) + C_2 \sinh(\sqrt{-\lambda}x)$$

  We see that $\phi(0) = C_1$ and $\phi'(0) = \sqrt{-\lambda}C_2$. Making $\phi(0) = \phi(2\pi)$ and $\phi'(0) = \phi'(2\pi)$ will give us a system of two equations in C_1, C_2. When we take the derivatives, every term will have $\sqrt{-\lambda}$, which can be canceled and leave us with:

$$\begin{array}{rl} C_1 \cosh(\sqrt{-\lambda}2\pi) + C_2 \sinh(\sqrt{-\lambda}2\pi) & = C_1 \\ C_1 \sinh(\sqrt{-\lambda}2\pi) + C_2 \cosh(\sqrt{-\lambda}2\pi) & = C_2 \end{array}$$

  There are multiple ways of proceeding; For example, if we multiply the first equation by C_2 and the second equation by $-C_1$, we get:

$$\begin{array}{rl} C_1C_2 \cosh(\sqrt{-\lambda}2\pi) + C_2^2 \sinh(\sqrt{-\lambda}2\pi) & = C_1C_2 \\ C_1^2 \sinh(\sqrt{-\lambda}2\pi) + C_1C_2 \cosh(\sqrt{-\lambda}2\pi) & = -C_1C_2 \\ \hline (C_1^2 + C_2^2) \sinh(\sqrt{-\lambda}2\pi) & = 0 \end{array}$$

  Either the constants are zero, or the hyperbolic sine is zero. The hyperbolic sine is zero only when $\lambda = 0$, so therefore, we only get the trivial solution.

- The last case is very similar: $\lambda > 0$. In this case, the solutions to the characteristic equation are complex, $r = \pm\sqrt{\lambda}i$, so the solution to the ODE is

$$\phi(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$$

  Therefore, $\phi(0) = \phi(2\pi)$ implies that:

$$C_1 = C_1 \cos(\sqrt{\lambda}2\pi) + C_2 \sin(\sqrt{\lambda}2\pi)$$

  Similarly, $\phi'(0) = \phi'(2\pi)$ implies that (this is simplified a bit- We canceled common factors):

$$C_2 = -C_1 \sin(\sqrt{\lambda}2\pi) + C_2 \cos(\sqrt{\lambda}2\pi)$$

There are multiple ways to proceed from here. One method would be to multiply the first equation by C_2 and the second equation by $-C_1$, then add the two:

$$\begin{array}{rl} C_1C_2 & = C_1C_2\cos(\sqrt{\lambda}2\pi) + C_2^2\sin(\sqrt{\lambda}2\pi) \\ -C_1C_2 & = C_1^2\sin(\sqrt{\lambda}2\pi) - C_2C_1\cos(\sqrt{\lambda}2\pi) \\ \hline 0 & = (C_1^2 + C_2^2)\sin(\sqrt{\lambda}2\pi) \end{array}$$

Therefore, to get a nontrivial solution to the ODE, we must have

$$\sqrt{\lambda}2\pi = n\pi \text{ for } n = 1, 2, 3, \cdots \quad \Rightarrow \quad \lambda = \frac{n^2}{4}, \text{ for } n = 1, 2, 3, \cdots$$

SIDE NOTE: There is a subtle bit to the remaining. That is,

$$C_1 = C_1\cos(n\pi) \qquad C_2 = C_2\cos(n\pi)$$

which only holds if n is even. Otherwise, we must have $C_1 = C_2 = 0$. Therefore, the only non-trivial solutions we get is when

$$\lambda = \frac{n^2}{4} \quad \text{for } n \text{ even}$$

However, it was fine for now if you stopped after determining λ.

23. Give the general solution to the Euler equation:

    (a) $x^2y'' + 4xy' + 2y = 0$
    SOLUTION: The ansatz is $y = x^k$, so substituting it into the DE gives us the characteristic equation:

    $$x^2k(k-1)x^{k-2} + 4xkx^{k-1} + 2x^k = 0 \quad \Rightarrow$$

    $$x^k(k(k-1) + 4k + 2) = 0 \quad \Rightarrow \quad k^2 + 3k + 2 = 0 \quad \Rightarrow \quad (k+1)(k+2) = 0$$

    The solution is: $y(x) = C_1x^{-1} + C_2x^{-2}$.

    (b) $x^2y'' - 3xy' + 4y = 0$
    SOLUTION: The ansatz is $y = x^k$, so substituting it into the DE gives us the characteristic equation:

    $$x^2k(k-1)x^{k-2} - 3xkx^{k-1} + 4x^k = 0 \quad \Rightarrow \quad x^k(k(k-1) - 3k + 4) = 0$$

    $$\Rightarrow \quad k^2 - 4k + 4 = 0 \quad \Rightarrow \quad (k-2)^2 = 0$$

    This is a repeated root, $k = 2$. Therefore, one solution is x^2 and the other is $x^2\ln(x)$, to get the general solution:

    $$y(x) = x^2(C_1 + C_2\ln(x)$$

24. For each PDE or boundary condition below, state whether or not it is LINEAR, and whether or not it is HOMOGENEOUS.

   (a) $u_x(0,t) = -H(u(0,t) - 30)$
   SOLUTION: Think about the equation as:

   $$Hu(0,t) + u_x(0,t) = 30H$$

   Therefore, it is linear in u and non-homogeneous.

   (b) $u_t(x,t) = u_x(x,t)u(x,t)$
   SOLUTION: This is nonlinear because u_x and u are multiplied together. It is homogeneous (bring everything with u over to the left side, and you're left with 0).

   (c) $u(0,t) + u_x(0,t) = 0$
   SOLUTION: This is linear and homogenous.

25. For each PDE, try using separation of variables to transform the equation into two ODEs (if possible). Do not solve the ODEs:

   (a) $xu_{xx} + u_t = 0$
   SOLUTION: With $u = XT$, you should find that we can get:

   $$x\frac{X''}{X} = -\frac{T'}{T}$$

   which separates the variables.

   (b) $u_{xx} + (x+y)u_{yy} = 0$
   SOLUTION: In this case, you should find something like, with $u = XY$, we get:

   $$\frac{X''}{X} + (x+y)\frac{Y''}{Y} = 0$$

   and we won't be able to separate x and y.