Review Questions, Final Exam

These questions only cover the final third of the course (since the second exam). For earlier material, see the previous review sheets.

Be sure you have gone through the homework problems from this third as well!

1. Solve each of the first order PDEs below.

   (a) $u_t - 3u_x = 0$ with $u(x,0) = \cos(x)$.

   (b) $u_t + xu_x = 1$ with $u(x,0) = f(x)$.

   (c) $u_t + 3tu_x = u$ with $u(x,0) = f(x)$

2. Questions about the Bessel functions $J_m(z)$ and $Y_m(z)$:

   (a) What ODE does J_m and Y_m solve?

   (b) As $z \to 0$, is there a limit for J_m and Y_m?

   (c) Going back to our spatial second order equation in ϕ, we can put that in SL form so that we have orthogonal functions.

   Be more specific about this- What integral is equal to zero?

3. Given the wave equation, verify d'Alembert's solution if:

$$u_{tt} = c^2 u_{xx} \qquad u(x,0) = \psi(x) \quad u_t(x,0) = 0$$

   Hint: Begin by assuming the solution is of the form $u(x,t) = f(x-ct) + g(x+ct)$.

4. Solve:

$$u_t = ku_{xx} + x \qquad 0 < x < L$$

   subject to the boundary conditions: $u_x(0,t) = t$, $u_x(L,t) = t^2$, and the initial condition $u(x,0) = f(x)$. You may leave the coefficients in integral form (you don't need to evaluate the integrals).

5. Find a formula for the coefficient a_{nm} if

$$f(x,y) \sim \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} a_{nm} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{H}\right)$$

   You may assume $0 < x < L$ and $0 < y < H$.

6. (a) Show that if u, v are eigenfunctions which both satisfy the same homogeneous boundary conditions:

$$\beta_1 \phi + \beta_2 \nabla\phi \cdot \vec{n} = 0$$

   then

$$\oint (u\nabla v - v\nabla u) \cdot \vec{n}\, ds = 0$$

(b) Let the operator be $L = \nabla^2$. Use the previous answer and Green's formula to show that, if ϕ_1, ϕ_2 are eigenfunctions for distinct eigenvalues, then ϕ_1, ϕ_2 are orthogonal. Note that the eigenfunctions both satisfy the same homogeneous boundary conditions.

7. Solve the Helmholtz equation

$$\nabla^2\phi + \lambda\phi = 0, \qquad [0,1] \times [0,1/4]$$

subject to:

$$\phi(0,y) = 0 \qquad \phi(x,0) = 0 \qquad \phi_x(1,y) = 0 \qquad \phi_y(x,1/4) = 0$$

8. Solve the heat equation on a disk with zero boundary conditions and initial condition $\alpha(r,\theta)$.

9. Solve Laplace's equation on a box in 3d, with

$$0 \le x \le L, \qquad 0 \le y \le L, \qquad 0 \le z \le W$$

with boundary conditions:

$$u_x(0,y,z) = 0 \qquad u_x(L,y,z) = 0 \qquad u_y(x,0,z) = 0 \qquad u_y(x,L,z) = 0$$

and the final two:

$$u_z(x,y,W) = 0 \qquad u_x(x,y,0) = 4\cos\left(\frac{3\pi x}{L}\right)\cos\left(\frac{4\pi y}{L}\right)$$

10. Solve

$$\begin{array}{rl} \text{PDE} & u_t = u_{xx} + 1 \qquad 0 < x < 1 \\ \text{BCs} & u_x(0,t) = 2 \qquad u(1,t) = 0 \\ \text{ICs} & u(x,0) = 2x - 1 \end{array}$$