Review Solutions, Final Exam

1. Solve the first order DEs:

   (a) $u_t - 3u_x = 0$ with $u(x, 0) = \cos(x)$.
   SOLUTION: We compare this with:

$$\frac{du}{dt} = u_t + \frac{dx}{dt}u_x = 0$$

   so that u solves the PDE if, on the curve defined by $x(t)$ (the characteristic curve), we have:

$$\frac{dx}{dt} = -3 \qquad \text{and} \qquad \frac{du}{dt} = 0$$

   so $u(x(t), t)$ is constant on the curve, and the curve is: $x = -3t + x_0$. Therefore, solving for $x_0 = x + 3t$, and with $u(x(0), 0) = cos(x_0)$, we get our full solution:

$$u(x, t) = \cos(x + 3t)$$

   (b) $u_t + xu_x = 1$ with $u(x, 0) = f(x)$.
   SOLUTION: Same idea as the last problem, where

$$\frac{dx}{dt} = x \qquad \frac{du}{dt} = 1$$

   From the first equation,

$$\frac{1}{x}\,dx = dt \quad \Rightarrow \quad \ln(x) = t + c \quad \Rightarrow \quad x = Ae^t = x_0 e^t$$

   These are the characteristic curves. Along these curves, $du/dt = 1$, so that $u = t + c$, so that

$$u(x(0), 0) = 0 + c = 0 + f(x_0) \quad \Rightarrow \quad u(x(t), t) = t + f(xe^{-t})$$

   NOTE: You can double-check yourself by verifying that this function does indeed solve the PDE.

   (c) $u_t + 3tu_x = u$ with $u(x, 0) = f(x)$.
   SOLUTION:

$$\frac{dx}{dt} = 3t \qquad \frac{du}{dt} = u$$

   From the first equation, $x = \frac{3}{2}t^2 + x_0$, and from the second equation, $u = Ae^t$. From this, we see that

$$u(x(0), 0) = A = f(x_0) \quad \Rightarrow \quad u(x, t) = f\left(x - \frac{3}{2}t^2\right)e^t$$

2. Questions about the Bessel functions $J_m(z)$ and $Y_m(z)$:

   (a) What ODE does J_m and Y_m solve?

   SOLUTION: These are solutions to the Bessel Equation of order m, given by (y is a functions of z):

   $$z^2y'' + zy' + (z^2 - m^2)y = 0$$

   (b) As $z \to 0$, is there a limit for J_m and Y_m?

   SOLUTION: The J_m are bounded at zero- In fact, $J_0(0) = 1$ and all others are 0. The other function Y_m becomes unbounded at $z = 0$.

   (c) Going back to our spatial second order equation in ϕ, we can put that in SL form so that we have orthogonal functions.

   Be more specific about this- What integral is equal to zero?

   SOLUTION: We want the SL form so we can get the weighting function. That would be:

   $$r(r\phi')' + (\lambda r^2 - m^2)\phi = 0 \quad \Rightarrow \quad (r\phi')' - \frac{m^2}{r}\phi = -\lambda r\phi$$

   Therefore, for m fixed,

   $$\int_0^a J_m(\sqrt{\lambda_{mn_1}}r)J_m(\sqrt{\lambda_{mn_2}}r)\, r\, dr = 0$$

3. D'Alembert's solution: Not on the exam.

4. Solve:

   $$u_t = ku_{xx} + x \qquad 0 < x < L$$

   subject to the boundary conditions: $u_x(0, t) = t$, $u_x(L, t) = t^2$, and the initial condition $u(x, 0) = f(x)$.

   SOLUTION: First we find a helper function to solve the boundary conditions, then we use the method of eigenfunctions. So, let $u = v + w$, and let dw/dx be linear, running from t to t^2. That gives us:

   $$w = tx + \frac{1}{2L}(t^2 - t)x^2$$

   And with $u = v + w$, we rewrite the expressions:

   $$u_t = v_t + \left(x + \frac{1}{2L}(2t - 1)x^2\right) \qquad u_{xx} = v_{xx} + \frac{1}{L}(t^2 - t)$$

   so that $u_t = ku_{xx} + x$ becomes

   $$v_t = kv_{xx} + S(x, t)$$

where

$$S(x,t) = x + kw_{xx}(x,t) - w_t(x,t)$$

which doesn't simplify by much, so we'll leave it as S. Now, we have zero boundary conditions, so our eigenvalues/functions are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \phi_n(x) = \cos\left(\frac{n\pi x}{L}\right)$$

Now we express the general solution as:

$$v(x,t) = \sum_{n=0}^{\infty} A_n(t)\cos\left(\frac{n\pi x}{L}\right)$$

where

$$v(x,0) = u(x,0) - w(x,0) = f(x) - 0 = f(x) = \sum_{n=0}^{\infty} A_n(0)\cos\left(\frac{n\pi x}{L}\right)$$

so that

$$A_0(0) = \frac{1}{L}\int_0^L f(x)\,dx \qquad \text{and} \qquad A_n(0) = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$$

We also can express S as a series:

$$S(x,t) = \sum_{n=0}^{\infty} q_n(t)\cos\left(\frac{n\pi x}{L}\right)$$

so that

$$q_0(t) = \frac{1}{L}\int_0^L S(x,t)\,dx \qquad \text{and} \qquad q_n(t) = \frac{2}{L}\int_0^L S(x,t)\cos\left(\frac{n\pi x}{L}\right)$$

Now we substitute the series for v into the PDE, and we get the infinite system of first order ODEs:

$$\sum_{n=0}^{\infty} \left(A_n'(t) + k\lambda A_n(t) - q_n(t)\right)\cos\left(\frac{n\pi x}{L}\right) = 0$$

so that, for $n = 0, 1, 2, 3, \ldots,$

$$A_n'(t) + k\lambda_n A_n(t) = q_n(t)$$

so that

$$A_n(t) = A_n(0)\mathrm{e}^{-k\lambda_n t} + \mathrm{e}^{-k\lambda_n t}\int_0^t q_n(\tau)\mathrm{e}^{k\lambda_n \tau}\,d\tau$$

So finally,

$$u(x,t) = \sum_{n=0}^{\infty} A_n(t)\cos\left(\frac{n\pi x}{L}\right) + tx + \frac{1}{2L}(t^2 - t)x^2$$

5. Find a formula for the coefficient a_{nm} if

$$f(x,y) \sim \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} a_{nm} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)$$

SOLUTION: You can write the integrals directly, or you can break them up. Notice that we need to treat $m = 0$ differently than the rest of the coefficients.

If you want to step through the computations, one way might be:

$$f(x,y) = \sum_{m=0}^{\infty}\left[\sum_{n=1}^{\infty} a_{nm}\sin\left(\frac{n\pi x}{L}\right)\right]\cos\left(\frac{m\pi y}{H}\right) = \sum_{m=0}^{\infty} F_m(x)\cos\left(\frac{m\pi y}{H}\right)$$

This is a cosine series, so:

$$F_0(x) = \frac{1}{H}\int_0^H f(x,y)\,dy \qquad \text{and} \qquad F_m(x) = \frac{2}{H}\int_0^H f(x,y)\cos\left(\frac{n\pi y}{L}\right)\,dy$$

Furthermore, since $F_m(x)$ is a sine series, we have:

$$a_{nm} = \frac{2}{L}\int_0^L F_m(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

You may leave your answer in that form, or if you prefer, you can write your answer in two dimensional form (with or without the previous stuff using $F_m(x)$):

$$a_{n0} = \frac{2}{LH}\int_0^L\int_0^H f(x,y)\sin\left(\frac{n\pi x}{L}\right)\,dy\,dx$$

and the others:

$$a_{nm} = \frac{4}{LH}\int_0^L\int_0^H f(x,y)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\,dy\,dx$$

6. (a) Show that if u, v are eigenfunctions which both satisfy the same homogeneous boundary conditions:

$$\beta_1\phi + \beta_2\nabla\phi\cdot\vec{n} = 0$$

then

$$\oint (u\nabla v - v\nabla u)\cdot\vec{n}\,ds = 0$$

SOLUTION: Substitute

$$\nabla v\cdot\vec{n} = -\frac{\beta_1}{\beta_2}v \qquad \text{and} \qquad \nabla u\cdot\vec{n} = -\frac{\beta_1}{\beta_2}u$$

Then the integral becomes:

$$\oint -\frac{\beta_1}{\beta_2}uv + \frac{\beta_1}{\beta_2}uv\,ds = 0$$

(This is related to showing that an operator is self-adjoint).

(b) Let the operator be $L = \nabla^2$. Use the previous answer and Green's formula to show that, if ϕ_1, ϕ_2 are eigenfunctions for distinct eigenvalues, then ϕ_1, ϕ_2 are orthogonal. Note that the eigenfunctions both satisfy the same homogeneous boundary conditions.

SOLUTION: Green's formula states that

$$\iint_R u\nabla^2 v - v\nabla^2 u\, dx\, dy = \oint (u\nabla v - v\nabla u)\, ds$$

Now, if we choose $u = \phi_{\lambda_1}$ and $v = \phi_{\lambda_2}$, which are eigenfunctions to $\nabla^2\phi = -\lambda\phi$, then:

$$\iint_R \phi_{\lambda_1}\nabla^2\phi_{\lambda_2} - \phi_{\lambda_2}\nabla^2\phi_{\lambda_1}\, dx\, dy = \iint_R -\lambda_2\phi_{\lambda_1}\phi_{\lambda_2} + \lambda_1\phi_{\lambda_2}\phi_{\lambda_1}\, dx\, dy =$$

$$(\lambda_1 - \lambda_2)\iint_R \phi_{\lambda_1}\phi_{\lambda_2}\, dx\, dy$$

And, from part (a) and Green's Theorem, we have:

$$\iint_R \phi_{\lambda_1}\nabla^2\phi_{\lambda_2} - \phi_{\lambda_2}\nabla^2\phi_{\lambda_1}\, dx\, dy = \oint (\phi_{\lambda_1}\nabla\phi_{\lambda_2} - \phi_{\lambda_2}\nabla\phi_{\lambda_1})\cdot\vec{n}ds = 0$$

Putting these two pieces together, we have

$$(\lambda_1 - \lambda_2)\iint_R \phi_{\lambda_1}\phi_{\lambda_2}\, dx\, dy = 0$$

so either $\lambda_1 = \lambda_2$ (which is not true), or the eigenfunctions are orthogonal.

7. Solve the Helmholtz equation

$$\nabla^2\phi + \lambda\phi = 0, \qquad [0, 1] \times [0, 1/4]$$

subject to:

$$\phi(0, y) = 0 \qquad \phi(x, 0) = 0 \qquad \phi_x(1, y) = 0 \qquad \phi_y(x, 1/4) = 0$$

SOLUTION: Separate variables so that $\phi(x, y) = XY$, and

$$X''Y + XY'' = -\lambda XY \quad \Rightarrow \quad \frac{X''}{X} = -\lambda - \frac{Y''}{Y} = -\mu$$

Therefore,

$$\begin{array}{cc} X'' & = -\mu X \\ X(0) = 0 & X'(1) = 0 \end{array} \qquad \frac{Y''}{Y} = -(\lambda - \mu) = -\tau$$

so that

$$Y'' = -\tau Y, \quad Y(0) = 0 \quad Y'(1/4) = 0$$

Now, you don't have to go into a lot of detail, but you should at least mention that in each case (for X and Y), the constants μ and tau are both positive.

Now for X, the first BC means that we have a sine function. The second BC means:

$$X'(1) = C_2\sqrt{\mu}\cos(\sqrt{\mu}) = 0$$

Therefore,

$$\mu_n = \left(\frac{(2n-1)\pi}{2}\right)^2 \qquad X_n = \sin\left(\frac{(2n-1)\pi}{2}x\right)$$

and in a similar vein,

$$\tau_n = [(4m-2)\pi]^2 \qquad Y_m = \sin\left((4m-2)\pi y\right)$$

so that

$$\lambda_{mn} = \left(\frac{(2n-1)\pi}{2}\right)^2 + [(4m-2)\pi]^2$$

and

$$\phi_{mn}(x,y) = \sin\left(\frac{(2n-1)\pi}{2}x\right)\sin\left((4m-2)\pi y\right)$$

8. Solve the heat equation on a disk with zero boundary conditions and initial condition $\alpha(r,\theta)$.

SOLUTION: Separate variables, since everything is homogeneous.

$$\begin{array}{lll} \text{PDE} & u_t = k\nabla^2 u & \\ \text{BCs} & u(a,\theta,t) = 0 & |u(0,\theta,t)| < \infty \\ \text{ICs} & u(r,\theta,0) = \alpha(r,\theta) & \end{array}$$

We'll have our now usual familiar separation, yielding Bessel functions in the radius. Now let $u(r,\theta,t) = f(r)g(\theta)h(t)$, and

$$fgh' = k\left[\frac{1}{r}\frac{\partial}{\partial r}(rf'gh) + \frac{1}{r^2}fg''h\right]$$

Divide by $kfgh$ and we get:

$$\frac{h'}{kh} = \frac{1}{rf}(rf')' + \frac{1}{r^2}\frac{g''}{g} = -\lambda$$

Therefore, in time, we get $h' = -k\lambda h$, or $h = \mathrm{e}^{-k\lambda t}$.

Continuing, we multiply by r^2, and write:

$$\frac{g''}{g} = -\lambda r^2 - \frac{r}{f}(rf')' = -\mu$$

so that

$$g'' = -\mu g$$

and finally we get our Bessel equation in f:

$$r^2 f'' + r f' + (\lambda r^2 - \mu) f = 0$$

so we take $mu = m^2$, and the solution is

$$f_m = J_m(\sqrt{\lambda} r)$$

With the BC, we have

$$J_m(\sqrt{\lambda} a) = 0 \quad \Rightarrow \quad \lambda = \left(\frac{z_{mn}}{a}\right)^2$$

where z_{mn} is the n^{th} zero of the Bessel function of order m. The eigenfunction is

$$f_{mn} = J_m(\sqrt{\lambda_{mn}} r)$$

Now we go to the angular equation:

$$g'' = -m^2 g \quad \Rightarrow \quad g(\theta) = c_1 \cos(m\theta) + c_2 \sin(m\theta) = c_1 g_{1m} + c_2 g_{2m}$$

Now, in shorthand, we might write:

$$u(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} f_{mn}(r) h_{mn}(t) (A_{mn} g_{1m} + B_{mn} g_{2m})$$

We can compute the coefficients using the initial condition

$$\alpha(r, \theta) = \sum_{m=0}^{\infty} \left[\sum_{n=1}^{\infty} A_{mn} J_m(\sqrt{\lambda_{mn}} r) \right] \cos(m\theta) + \left[\sum_{n=1}^{\infty} B_{mn} J_m(\sqrt{\lambda_{mn}} r) \right] \sin(m\theta)$$

Now finish this up like we did above in Problem 5.

9. Solve Laplace's equation on a box in 3d, with

$$0 \le x \le L, \qquad 0 \le y \le L, \qquad 0 \le z \le W$$

with boundary conditions:

$$u_x(0, y, z) = 0 \qquad u_x(L, y, z) = 0 \qquad u_y(x, 0, z) = 0 \qquad u_y(x, L, z) = 0 \qquad u_z(x, y, W) = 0$$

and

$$u_z(x, y, 0) = 4 \cos\left(\frac{3\pi x}{L}\right) \cos\left(\frac{4\pi y}{L}\right)$$

SOLUTION: Let $u = XYZ$. Substituting into Laplace's equation gives

$$\frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} = -\lambda$$

splitting off X we have:

$$\begin{array}{c} X'' + \lambda X = 0 \\ X'(0) = 0 \quad X'(L) = 0 \end{array} \qquad\qquad \frac{Y''}{Y} + \frac{Z''}{Z} = \lambda$$

Continuing in Y, Z we have

$$\frac{Y''}{Y} = \lambda - \frac{Z''}{Z} = -\mu \quad \Rightarrow \quad \begin{array}{c} Y'' + \mu Y = 0 \\ Y'(0) = 0 \quad Y'(L) = 0 \end{array} \quad \text{and} \quad \frac{Z''}{Z} = \lambda + \mu = \tau$$

Therefore, lastly we have

$$\begin{array}{c} Z'' - \tau Z = 0 \\ Z'(W) = 0 \quad Z'(0) = f(x, y) \end{array}$$

The eigenvalues/functions for X, Y are the familiar ones.

$$\begin{array}{llllll} \lambda_m & = m^2\pi^2/L^2 & \mu_n & = n^2\pi^2/L^2 & \tau_{mn} & = \lambda_m + \mu_n \\ X_m & = \cos\left(\frac{m\pi x}{L}\right) & Y_n & = \cos\left(\frac{n\pi x}{L}\right) & Z_{mn} & = \cosh\left(\sqrt{\tau_{mn}}(W - z)\right) \end{array}$$

Now, we use some shorthand to write the full solution, and put in the initial condition:

$$u(x, y, z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn} X_m Y_n Z_{mn}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-A_{mn}\sqrt{\tau_{mn}}\sinh(\sqrt{\tau_{mn}}(W))\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right) = 4\cos\left(\frac{3\pi x}{L}\right)\cos\left(\frac{4\pi y}{L}\right)$$

Therefore, the only nonzero coefficient is A_{34}, and

$$A_{34} = \frac{-4}{\sqrt{\tau_{mn}}\sinh(\sqrt{\tau_{mn}}(W))}$$

10. Solve

$$\begin{array}{rll} \text{PDE} & u_t = u_{xx} + 1 & 0 < x < 1 \\ \text{BCs} & u_x(0, t) = 2 & u(1, t) = 0 \\ \text{ICs} & u(x, 0) = 2x - 1 & \end{array}$$

SOLUTION: First the helper function, then find the spatial eigenfunctions for a basis for solution space, then write everything in terms of that basis resulting in an infinite system of first order ODEs.

- The helper function is $w(x) = 2x - 2$.
  If we define $u = v + w$, then the PDE in v is:

$$\begin{array}{rll} \text{PDE} & v_t = v_{xx} + 1 & 0 < x < 1 \\ \text{BCs} & v_x(0,t) = 0 & v(1,t) = 0 \\ \text{ICs} & v(x,0) = (2x-1) - (2x-2) = 1 & \end{array}$$

- The spatial eigenvalues will solve the BVP:

$$X'' + \lambda X = 0 \qquad X'(0) = 0 \quad X(1) = 0$$

  There was a little algebra trick we've used previously that helps a bit. Since the function is zero at $x = 1$, we'll write the functions as:

$$X(x) = C_1 \cos(\sqrt{\lambda}(x-1)) + C_2 \sin(\sqrt{\lambda}(x-1))$$

  (or you could have used $(1-x)$ as well). Now it's a bit easier to get the coefficients, with $C_1 = 0$, and

$$X'(0) = C_2\sqrt{\lambda}\cos(\sqrt{\lambda}) = 0 \quad \Rightarrow \quad \lambda_n = \left(\frac{(2n-1)\pi}{2}\right)^2$$

  And we now have our basis functions as the following:

$$X_n(x) = \sin\left(\frac{(2n-1)\pi}{2}(x-1)\right)$$

- Now we assume

$$v(x,t) = \sum_{n=1}^{\infty} A_n(t) X_n(x)$$

  and substitute this into the PDE. Note that we'll also need the series expansion of 1:

$$1 = \sum_{n=1}^{\infty} q_n \sin\left(\frac{(2n-1)\pi}{2}(x-1)\right) \quad \Rightarrow \quad q_n = \frac{-4}{(2n-1)\pi}$$

  Now reverting back to our shorthand notation, put everything back into the PDE to get our system of ODEs:

$$\sum A_n'(t) X_n = \sum(-\lambda_n A_n) X_n + \sum q_n X_n$$

  so that

$$A_n'(t) + \lambda_n A_n(t) = q_n$$

  or

$$A_n(t) = \frac{q_n}{\lambda_n} + C\mathrm{e}^{-\lambda_n t}$$

What about $A_n(0)$? We use our initial condition for that-

$$v(x,0) = 1 = \sum A_n(0)X_n(x)$$

and we see that $A_n(0) = q_n = -4/(2(n-1)\pi)$, so that

$$A_n(t) = q_n \left(\frac{1}{\lambda_n} + \left(1 - \frac{1}{\lambda_n}\right) \mathrm{e}^{-\lambda_n t} \right)$$

Finally, we put it all back together:

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t)X_n(x) + 2x - 2$$