Review Questions: Exam 2

Exam 2 will cover Chapters 3-5 (See the class website for specific sections). A good way to review is to go back over the homework questions - especially the first couple of steps of each.

The exam will be 50 minutes in length, no calculators or notes allowed. You may assume things like the orthogonality of sines and cosines (see p 58, 63).

1. True or False? (And give a short answer)
   (a) If $f(x)$ is piecewise smooth on $[0, L]$, we can find a series representation using either a sine or a cosine series.
   (b) If $f(x)$ is piecewise smooth on $[-L, L]$, we can find a series representation using either a sine or a cosine series.
   (c) The sine series for $f(x)$ on $[-L, L]$ will converge to the odd extension of f.
   (d) The Gibbs phenomenon (an overshoot of the Fourier series) occurs only when we use a finite number of terms in the Fourier series to represent a function that is discontinuous.
   (e) The functions $\sin(nx)$ for $n = 1, 2, 3, \cdots$ are orthogonal to the functions $\cos(mx)$ for $m = 0, 1, 2, 3, \cdots$ on the interval $[0, \pi]$.

2. Short Answer:
   (a) Questions about when the Fourier series will be continuous:
      i. Let $-L \leq x \leq L$. For what functions f can we guarantee that the Fourier series of f will be continuous (at every real number)?
      ii. How does the previous answer change if we have $0 \leq x \leq L$ for f and use a Fourier cosine series?
      iii. How does the first answer change if we have $0 \leq x \leq L$ for f and use a Fourier sine series?
   (b) Let $f(x) = 3x + 5$. Compute the even and odd parts of f.
   (c) Differentiation and the Fourier series:
      i. Generally speaking, if f is defined on $[-L, L]$, under what conditions can we differentiate the general Fourier series to obtain the series for $f'(x)$?
      ii. Does our answer change if we use only a cosine series on $[0, L]$?
      iii. Does our answer change if we use only a sine series on $[0, L]$?

3. Let
$$f(x) = \begin{cases} 2x & \text{for } 0 < x < 1 \\ 2 & \text{for } 1 < x < 2 \end{cases}$$
   (a) Write the even extension of f as a piecewise defined function.
   (b) Write the odd extension of f as a piecewise defined function.
   (c) Draw a sketch of the periodic extension of f.
   (d) Find the Fourier sine series (FSS) for f, and draw the FSS on the interval $[-4, 4]$.
   (e) Find the Fourier cosine series (FCS) for f, and draw the FCS on the interval $[-4, 4]$.

4. Suppose that $0 \leq x \leq L$, and $f(x)$ is represented by the Fourier sine series,
$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$
   Then we know that $f'(x)$ has a Fourier cosine series,
$$f'(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

(a) If we differentiate the series for f term by term, what is another cosine series for $f'(x)$?

(b) Use integration by parts to show that

$$A_0 = \frac{1}{L}(f(L) - f(0))$$

and, for $n \neq 0$,

$$A_n = \frac{2}{L}\int_0^L f'(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$$

$$= \frac{2}{L}((-1)^n f(L) - f(0) + \frac{n\pi}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx)$$

$$= \frac{2}{L}((-1)^n f(L) - f(0)) + \frac{n\pi}{L}B_n$$

(c) Put (a), (b) together to get a formula for the series of the derivative of f,

$$f'(x) \sim \underline{\qquad\qquad\qquad} + \sum_{n=1}^{\infty} \underline{\qquad\qquad\qquad} \cos\left(\frac{n\pi x}{L}\right)$$

5. Consider $u_t = ku_{xx}$ subject to the conditions: $u_x(0,t) = 0$, $u_x(L,t) = 0$ and $u(x,0) = f(x)$.

   Solve in the following way: Look for solutions as a Fourier cosine series, and assume that u, u_x are continuous, and u_{xx}, u_t are PWS.

6. Solve the following nonhomogeneous problem:

$$u_t = ku_{xx} + \mathrm{e}^{-t} + \mathrm{e}^{-2t}\cos\left(\frac{3\pi x}{L}\right)$$

   where we have insulated ends at $x = 0$ and $x = L$, and $u(x,0) = f(x)$, and we can assume that $2 \neq k(3\pi/L)^2$. Use the following method: Look for the solution as a Fourier cosine series.

7. Let $f(x)$ be given as below.

$$f(x) = \begin{cases} x \text{ if } -1 < x < 0 \\ 1 + x \text{ if } 0 < x < 1 \end{cases}$$

   (a) Find the Fourier series for f (on $[-1,1]$), and draw a sketch of it on $[-3,3]$.

   (b) Find the Fourier sine series for f on $[0,1]$ and draw a sketch of it on $[-3,3]$.

   (c) Find the Fourier cosine series for f on $[0,1]$ and draw a sketch of it on $[-3,3]$.

8. Put the folowing BVP in Sturm-Liouville form:

$$(1-x^2)\phi'' - 2x\phi' + (1+\lambda x)y = 0 \qquad \phi(-1) = 0 \quad \phi(1) = 0$$

   on the interval $-1 < x < 1$.

9. Given the differential equation: $\phi'' + \lambda\phi = 0$, determine the eigenvalues λ and eigenfunctions ϕ if ϕ satisfies the following boundary conditions (analyze all three cases; you may assume the eigenvalues are real).

   (a) $\phi(a) = 0$, $\phi(b) = 0$

   (b) $\phi'(0) = 0$ and $\phi'(L) = 0$

   (c) $\phi(0) = 0$ and $\phi'(L) = 0$

10. Solve

$$u_{tt} = \frac{4}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \qquad 0 < r < 1, t > 0$$

with $u(r,0) = f(r)$, $u(0,t)$ bounded and $u(1,t) = 0$. You should assume that the (radial) eigenfunctions are known and complete.

11. Given the BVP in regular S-L form, with appropriate boundary conditions, show that the eigenfunctions corresponding to two distinct eigenvalues are orthogonal with respect to $\sigma(x)$. Hint: Consider

$$\int_a^b \phi_n L(\phi_m) - \phi_m L(\phi_n)\, dx$$

12. Solve using separation of variables:

$$\begin{array}{ll} \text{PDE} & u_{tt} = u_{xx} \qquad 0 < x < 1, t > 0 \\ \text{BCs} & u(0,t) = 0 \quad u(1,t) = 0 \\ \text{ICs} & u(x,0) = \sin(\pi x) + \frac{1}{2}\sin(3\pi x) + 3\sin(7\pi x) \\ & u_t(x,0) = \sin(2\pi x) \end{array}$$

(Keep the constants with the spatial equation)

13. Consider $L(\phi) = \phi''''$. Find an expression like Green's Formula for this operator on $0 < x < 1$. HINT: Use integration by parts for $\int_0^1 uL(v)\, dx$ until you get $\int_0^1 vL(u)\, dx$ on the right side of the equation. That is, you should have something in the form:

$$\int_0^1 uL(v)\, dx = (\ldots) + \int_0^1 vL(u)\, dx$$

14. Consider

$$\begin{aligned} \rho u_{tt} &= T_0 u_{xx} + \alpha u \\ u(0,t) = u(L,t) &= 0 \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}$$

with $\rho(x) > 0$, $\alpha(x) < 0$, and T_0 constant.

Assume that the appropriate eigenfunctions (in space) are known. Solve the PDE using separation of variables.

15. Use the Rayleigh quotient to obtain a reasonably accurate upper bound for the lowest eigenvalue of

$$\phi'' + (\lambda - x)\phi = 0 \qquad \phi'(0) = 0 \quad \phi'(1) + 2\phi(1) = 0$$

16. Use the alternate form of the Rayleigh quotient below to compute $R(u)$, if λ_n are the eigenvalues, ϕ_n the eigenfunctions, and $u = 2\phi_1 + 3\phi_2$. To simplify our computations, you may assume that the eigenfunctions have been normalized so that $\int_a^b \phi_n^2 \sigma(x)\, dx = 1$. In that case, the Rayleigh quotient simplifies to:

$$R(\phi) = -\int_a^b \phi L(\phi)\, dx$$

17. Suppose we define a linear operator as: $L(y) = y'$, where y satisfies the BCs $y(0) - 3y(1) = 0$. Find an expression for the adjoint operator L^* so that

$$\langle u, L(v)\rangle = \langle L^*(u), v\rangle$$

You may assume that $\langle f, g\rangle = \int_0^1 f(t)g(t)\, dt$. You should also determine the BCs for functions used by L^*.