Review SOLUTIONS: Exam 2

1. True or False? (And give a short answer)

   (a) If $f(x)$ is piecewise smooth on $[0, L]$, we can find a series representation using either a sine or a cosine series.

   SOLUTION: TRUE. If we use a sine series, the series will converge to the odd extension of f on $[-L, L]$, then to the periodic extension of that over the reals (with the usual caveat about points at which the periodic extension has a jump discontinuity).

   If we use a cosine series (with the constant term), the series will converge to the even extension of f on $[-L, L]$, then to the periodic extension of that over the reals (again with the caveat about the jump discontinuities).

   (b) If $f(x)$ is piecewise smooth on $[-L, L]$, we can find a series representation using either a sine or a cosine series.

   SOLUTION: FALSE. On the full interval $[-L, L]$ we need both sines and cosines to get a *complete* set of functions (that is, both sines and cosines are needed to make a basis for this vector space). In other words, we must assume the form:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

   (Note that we're assume on these problems that the arguments for the sine and cosine have $n\pi x/L$ in them... Without that assumption, the statement could be true).

   (c) The sine series for $f(x)$ on $[-L, L]$ will converge to the odd extension of f.

   SOLUTION: FALSE. The sine series for $f(x)$ will converge to the *odd part of* f, which is given by:

$$f_{\text{odd}} = \frac{1}{2}(f(x) - f(-x))$$

   If f itself was odd, the statement would be true (but TRUE would mean the statement is true for all f). A similar argument can be made about cosines and the even part of f.

   The key point of this question is to be sure you know the difference between the even/odd *part* of f and the even/odd *extension* of f.

   (d) The Gibbs phenomenon (an overshoot of the Fourier series) occurs only when we use a finite number of terms in the Fourier series to represent a function that is discontinuous.

   SOLUTION: TRUE. If we use an infinite number of terms, there is no "overshoot", and the series converges to $f(x)$ where f is continuous, and $\frac{1}{2}(f(x+) + f(x-))$ where f is discontinuous. Therefore, the only time that you can get this overshooting phenomenon is when you use a finite number of terms in the sum.

   (e) The functions $\sin(nx)$ for $n = 1, 2, 3, \cdots$ are orthogonal to the functions $\cos(mx)$ for $m = 0, 1, 2, 3, \cdots$ on the interval $[0, \pi]$.

   SOLUTION: FALSE. For example, consider the following integral of the product of $\sin(x)$ with 1 (or $\cos(0x)$):

$$\int_0^{\pi} \sin(x) \cdot 1\, dx = \cos(x)|_1^{\pi} = 2$$

   Similarly, (you wouldn't need to compute this without a table):

$$\int_0^{\pi} \sin(x)\cos(2x)\, dx = -\frac{2}{3}$$

   We should note, however, that if the interval is changed to $[-\pi, \pi]$, then the statement would have been TRUE.

2. Short Answer:

(a) Questions about when the Fourier series will be continuous:

i. Let $-L \leq x \leq L$. For what functions f can we guarantee that the Fourier series of f will be continuous (at every real number)?
SOLUTION: For the Fourier series to be continuous in the interior of $(-L, L)$, the function f must be as well. For the series to be continuous at every real number, the periodic extension of f must be continuous as well- Which means that $f(-L) = f(L)$.

ii. How does the previous answer change if we have $0 \leq x \leq L$ for f and use a Fourier cosine series?
SOLUTION: We require f to be continuous in $(0, L)$, and we need the even extension of f to be continuous at $x = 0$ (which it always is if f is continuous on $[0, L]$), then we need the periodic extension to be continuous on the reals- For the even extension, we always have $f(L) = f(-L)$. Therefore, in this case, the Fourier cosine series for f will be continuous at every real number as long as f is continuous on $[0, L]$.

iii. How does the first answer change if we have $0 \leq x \leq L$ for f and use a Fourier sine series?
SOLUTION: As usual, f must first be continuous on $(0, L)$. Then the odd extension needs to be continuous on $[-L, L]$. This occurs if $f(0) = 0$. The odd extension would then need to be continuous as a periodic extension, which only happens if $f(L) = 0$ as well.

(b) Let $f(x) = 3x + 5$. Compute the even and odd parts of f.
SOLUTION: The odd part is $f_{\text{odd}} = \frac{1}{2}(f(x) - f(-x)) = 3x$
The even part is $f_{\text{even}} = \frac{1}{2}(f(x) + f(-x)) = 5$
Side note: If we had the full Fourier series for $3x + 5$ on the interval $[-L, L]$, then the sine series would converge to $3x$ and the cosine series to 5 (in fact, the cosine series *is* just the number 5).

(c) Differentiation and the Fourier series:

i. Generally speaking, if f is defined on $[-L, L]$, under what conditions can we differentiate the general Fourier series to obtain the series for $f'(x)$?
SOLUTION: We need the Fourier series to be continuous everywhere (that means f is continuous on $[-L, L]$ and $f(-L) = f(L)$), and f' is PWS (which will guarantee the convergence of its Fourier series). Further, note that if f' is not continuous at a point x_0, the Fourier series for f' will converge, as usual, to $\frac{1}{2}(f'(x_0+) + f(x_0-)$

ii. Does our answer change if we use only a cosine series on $[0, L]$?
SOLUTION: The general part of the solution does not- That is, we need (i) the Fourier cosine series to converge to f (so f is PWS), (ii) the series should be be continuous everywhere (so in this case, f just needs to be continuous on $[0, L]$), and (iii) f' must be PWS so that we know it has a series representation.

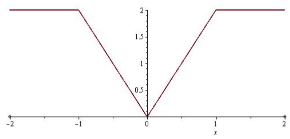
iii. Does our answer change if we use only a sine series on $[0, L]$?
SOLUTION: Again, the general statement does not change, just the conditions under which the statements will be true change. That is, we need (i) the Fourier sine series to converge to f (so f is PWS), (ii) the series should be be continuous everywhere (so in this case, f just needs to be continuous on $[0, L]$ AND $f(0) = 0, f(L) = 0$), and (iii) f' must be PWS so that we know it has a series representation.

3. Let

$$f(x) = \begin{cases} 2x & \text{for } 0 < x < 1 \\ 2 & \text{for } 1 < x < 2 \end{cases}$$

(a) Write the even extension of f as a piecewise defined function.
The even extension of f on the interval $[-2, 2]$ would be defined as:

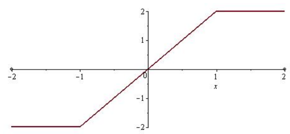
$$f(x) = \begin{cases} 2 & \text{for } -2 < x < -1 \\ -2x & \text{for } -1 < x < 0 \\ 2x & \text{for } 0 < x < 1 \\ 2 & \text{for } 1 < x < 2 \end{cases}$$



(b) Write the odd extension of f as a piecewise defined function.

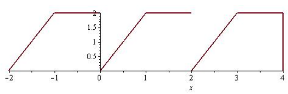
Similarly, the odd extension on $[-2, 2]$ is defined as:

$$f(x) = \begin{cases} -2 & \text{for } -2 < x < -1 \\ 2x & \text{for } -1 < x < 0 \\ 2x & \text{for } 0 < x < 1 \\ 2 & \text{for } 1 < x < 2 \end{cases}$$

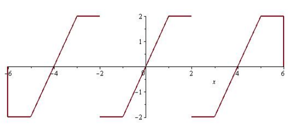


(c) Draw a sketch of the periodic extension of f.

SOLUTION:



(d) Find the Fourier sine series (FSS) for f, and draw the FSS on the interval $[-4, 4]$.



NOTE: The vertical lines don't belong in the graph, and in the places where there is a jump discontinuity (at -6, -2, 2, 6), we ought to draw a point to indicate that the series converges to zero there.

The algebraic form of the series is:

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) \quad \Rightarrow \quad b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx$$

Therefore, with $L = 2$:

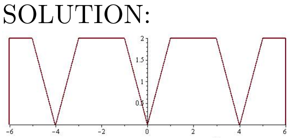
$$b_n = \int_0^1 2x \sin\left(\frac{n\pi x}{2}\right)\, dx + \int_1^2 2 \sin\left(\frac{n\pi x}{2}\right)\, dx =$$

$$-\frac{4}{n^2\pi^2}(-2\sin\left(\frac{n\pi}{2}\right) + n\pi\cos\left(\frac{n\pi}{2}\right) - \frac{4}{n\pi}(-1 + (-1)^n)$$

It is possible to simplify that a bit, but that is unnecessary for the exam.

(e) Find the Fourier cosine series (FCS) for f, and draw the FCS on the interval $[-4, 4]$.

SOLUTION:



NOTE: The vertical lines don't belong in the graph, the series would continue out in a continuous fashion.

The algebraic form of the series is:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right) \quad \Rightarrow \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$$

The formula for a_0 is slightly different, so do that one first:

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx = \int_0^1 2x\,dx + \int_1^2 2\,dx = 3$$

And, for $n = 1, 2, 3, \ldots$:

$$a_n = \int_0^1 2x\cos\left(\frac{n\pi x}{2}\right)\,dx + \int_1^2 2\cos\left(\frac{n\pi x}{2}\right)\,dx =$$

$$\frac{4}{n^2\pi^2}(-2 + 2\cos\left(\frac{n\pi}{2}\right) + n\pi\sin\left(\frac{n\pi}{2}\right) - \frac{4}{n\pi}\sin\frac{n\pi}{2}$$

It is possible to simplify that a bit, but that is unnecessary for the exam.

4. Suppose that $0 \le x \le L$, and $f(x)$ is represented by the Fourier sine series,

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

Then we know that $f'(x)$ has a Fourier cosine series,

$$f'(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

(a) If we differentiate the series for f term by term, what is another cosine series for $f'(x)$?
SOLUTION: Another way of expressing f' should be

$$f'(x) \sim \sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right) B_n \cos\left(\frac{n\pi x}{L}\right)$$

(b) Use integration by parts to show that

$$A_0 = \frac{1}{L}(f(L) - f(0))$$

SOLUTION FOR A_0:

$$A_0 = \frac{1}{L}\int_0^L f'(x)\,dx \quad \Rightarrow \quad A_0 = \frac{1}{L}(f(L) - f(0))$$

Continuing:

$$A_n = \frac{2}{L}\int_0^L f'(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$$

$$= \frac{2}{L}((-1)^n f(L) - f(0) + \frac{n\pi}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx)$$

$$= \frac{2}{L}((-1)^n f(L) - f(0)) + \frac{n\pi}{L}B_n$$

SOLUTION: Easy to see if you build the table to do integration by parts.

(c) Put (a), (b) together to get a formula for the series of the derivative of f,

SOLUTION: This summarizes the formula- Given

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

Then the derivative has the series:

$$f'(x) \sim \frac{1}{L}(f(L) - f(0)) + \sum_{n=1}^{\infty} \frac{2}{L}((-1)^n f(L) - f(0)) + \frac{n\pi}{L} B_n \cos\left(\frac{n\pi x}{L}\right)$$

5. Consider $u_t = ku_{xx}$ subject to the conditions: $u_x(0,t) = 0$, $u_x(L,t) = 0$ and $u(x,0) = f(x)$.

Solve in the following way: Look for solutions as a Fourier cosine series, and assume that u, u_x are continuous, and u_{xx}, u_t are PWS.

SOLUTION: Let

$$u(x,t) = A_0(t) + \sum_{n=1}^{\infty} A_n(t) \cos\left(\frac{n\pi x}{L}\right)$$

We can differentiate in time as long as u is continuous and u_t is PWS:

$$u_t = A_0'(t) + \sum_{n=1}^{\infty} A_n'(t) \cos\left(\frac{n\pi x}{L}\right)$$

We're told that u_x is continuous and u_{xx} is PWS, so we can differentiate twice in x:

$$u_{xx} = -\sum_{n=1}^{\infty} \frac{n^2\pi^2}{L^2} A_n(t) \cos\left(\frac{n\pi x}{L}\right)$$

Now, $u_t = ku_{xx}$, so we can equate coefficients of the Fourier series. First, for $n = 0$:

$$A_0'(t) = 0 \quad \Rightarrow \quad A_0(t) = a_0$$

Similarly, for $n = 1, 2, 3, \ldots$:

$$A_n'(t) = -\frac{n^2\pi^2}{L^2} A_n(t) \quad \Rightarrow \quad A_n(t) = a_n \mathrm{e}^{-(n\pi/L)^2\, t}$$

so that

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \mathrm{e}^{-(n\pi/L)^2\, t} \cos\left(\frac{n\pi x}{L}\right)$$

Finally, we require that $u(x,0) = f(x)$, or:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

If we multiply both sides by 1, then integrate from $x = 0$ to $x = L$, we get (by orthogonality):

$$\int_0^L f(x)\,dx = a_0 \int_0^L dx + 0 + 0 + 0 + \ldots \quad \Rightarrow \quad a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

And if we multiply both sides by $\cos\left(\frac{k\pi x}{L}\right)$ and integrate, we get (again by orthogonality):

$$\int_0^L f(x) \cos\left(\frac{k\pi x}{L}\right)\,dx = 0 + 0 + \ldots + a_k \int_0^L \cos^2\left(\frac{k\pi x}{L}\right)\,dx + 0 + 0 + \ldots$$

(It's quicker just to recall that the integral on the left is $L/2$ than to go through the double angle formula- That's fine). Therefore,

$$a_k = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{k\pi x}{L}\right)\,dx$$

6. Solve the following nonhomogeneous problem:

$$u_t = ku_{xx} + \mathrm{e}^{-t} + \mathrm{e}^{-2t}\cos\left(\frac{3\pi x}{L}\right)$$

where we have insulated ends at $x = 0$ and $x = L$, and $u(x, 0) = f(x)$, and we can assume that $2 \neq k(3\pi/L)^2$. Use the following method: Look for the solution as a Fourier cosine series.

SOLUTION: We have the eigenvalues and eigenfunctions:

$$\lambda = \frac{n^2\pi^2}{L^2} \qquad \phi_n(x) = \cos\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \cdots$$

Therefore, we assume solutions to the non-homogeneous equation are in the form:

$$u(x, t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t)\cos\left(\frac{n\pi x}{L}\right)$$

We also note that we can write the function $q(x, t) = \mathrm{e}^{-t} + \mathrm{e}^{-2t}\cos\left(\frac{3\pi x}{L}\right)$ as a cosine series:

$$\mathrm{e}^{-t} + \mathrm{e}^{-2t}\cos\left(\frac{3\pi x}{L}\right) = q_0(t)\cos(0x) + q_1(t)\cos\left(\frac{\pi x}{L}\right) + q_2(t)\cos\left(\frac{2\pi x}{L}\right) + q_3(t)\cos\left(\frac{3\pi x}{L}\right) + \cdots$$

Therefore, we get $q_0(t) = \mathrm{e}^{-t}$, $q_3(t) = \mathrm{e}^{-2t}$, and $q_n(t) = 0$ for all other n.

Now, we substitute our series into the PDE. The prime notation is the derivative in time:

$$a_0'(t) + \sum_{n=1}^{\infty} a_n'(t)\cos\left(\frac{n\pi x}{L}\right) = -k\sum_{n=1}^{\infty} a_n(t)\left(\frac{n^2\pi^2}{L^2}\right)\cos\left(\frac{n\pi x}{L}\right) + \mathrm{e}^{-t} + \mathrm{e}^{-2t}\cos\left(\frac{3\pi x}{L}\right)$$

This leads to the (infinite) system of ODEs, one for each n, which we will also solve:

- $n = 0$:

  $$a_0'(t) = \mathrm{e}^{-t} \quad \Rightarrow \quad a_0(t) = C - \mathrm{e}^{-t}$$

  We can write the solution in terms of $a_0(0)$ as they do in the book:

  $$a_0(0) = C - 1 \quad \Rightarrow \quad C = 1 + a_0(0)$$

  Therefore,

  $$a_0(t) = 1 + a_0(0) - \mathrm{e}^{-t}$$

- For $n = 3$, we get something similar. Use an integrating factor to solve:

  $$a_3'(t) = -\frac{9\pi^2 k}{L^2}a_3(t) + \mathrm{e}^{-2t}$$

  $$\left(a_3(t)\mathrm{e}^{(9\pi^2 k/L^2)t}\right)' = \mathrm{e}^{-2t}\mathrm{e}^{(9\pi^2 k/L^2)t} = \mathrm{e}^{(-2+9\pi^2 k/L^2)t}$$

  Antidifferentiating,

  $$a_3(t)\mathrm{e}^{(9\pi^2 k/L^2)t} = \frac{1}{(-2 + 9\pi^2 k/L^2)}\mathrm{e}^{(-2+9\pi^2 k/L^2)t} + C$$

This is where we need to be sure that $2 \neq 9\pi^2 k/L^2$ (if this was true, the right side would reduce to 1). Simplifying, we get:

$$a_3(t) = \frac{1}{(-2 + 9\pi^2 k/L^2)} \mathrm{e}^{-2t} + C\mathrm{e}^{(9\pi^2 k/L^2)t}$$

We can express this in terms of $a_3(0)$ as before:

$$a_3(0) = \frac{1}{(-2 + 9\pi^2 k/L^2)} + C \quad \Rightarrow \quad C = a_3(0) - \frac{1}{(-2 + 9\pi^2 k/L^2)}$$

Therefore,

$$a_3(t) = \frac{1}{(-2 + 9\pi^2 k/L^2)} \mathrm{e}^{-2t} + \left(a_3(0) - \frac{1}{(-2 + 9\pi^2 k/L^2)} \right) \mathrm{e}^{(9\pi^2 k/L^2)t}$$

- For all other n:

$$a_n'(t) = -\frac{n^2 \pi^2 k}{L^2} a_n(t)$$

The solution for each of these is:

$$a_n(t) = a_n(0)\mathrm{e}^{-(n^2\pi^2 k/L)t}$$

7. Let $f(x)$ be given as below.

$$f(x) = \begin{cases} x \text{ if } -1 < x < 0 \\ 1 + x \text{ if } 0 < x < 1 \end{cases}$$

(a) Find the Fourier series for f (on $[-1, 1]$), and draw a sketch of it on $[-3, 3]$.

SOLUTION: I'll leave the sketch to you. The main purpose here is to have you recall the formulas for the series coefficients. In this case,

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

with the formulas:

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx = \frac{1}{2} \left(\int_{-1}^{0} x\, dx + \int_{0}^{1} (1 + x)\, dx \right) = \frac{1}{2}$$

and,

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(n\pi x/L)\, dx = \int_{-1}^{0} x \cos(n\pi x)\, dx + \int_{0}^{1} (1 + x) \cos(n\pi x)\, dx = 0$$

and similarly

$$b_n = \frac{1}{L} \int_{-1}^{1} f(x) \sin(n\pi x/L)\, dx = \int_{-1}^{0} x \sin(n\pi x)\, dx + \int_{0}^{1} (1 + x) \sin(n\pi x)\, dx =$$

$$\frac{1}{n\pi}(1 - (-1)^n - 2(-1)^n)$$

(NOTE: If you subtracted 1/2 from your function $f(x)$, it becomes an odd function- That's why the cosine terms ended up being zero).

(b) Find the Fourier sine series for f on $[0,1]$ and draw a sketch of it on $[-3,3]$.
SOLUTION: Again, the main point here is to have you recall the formulas and set up the integrals:

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

with

$$b_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\,dx = 2\int_0^1 (1+x)\sin(n\pi x)\,dx = 2\frac{1-2(-1)^n}{n\pi}$$

(c) Find the Fourier cosine series for f on $[0,1]$ and draw a sketch of it on $[-3,3]$.
SOLUTION: The formulas:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

with

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx = \int_0^1 (1+x)\,dx = \frac{3}{2}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos(n\pi x/L)\,dx = 2\int_0^1 (1+x)\cos(n\pi x)\,dx = 2\frac{(-1)+(-1)^n}{n^2\pi^2}$$

8. Put the folowing BVP in Sturm-Liouville form:

$$(1-x^2)\phi'' - 2x\phi' + (1+\lambda x)\phi = 0 \qquad \phi(-1)=0 \quad \phi(1)=0$$

on the interval $-1 < x < 1$.

TYPO: The y in the equation should have been ϕ (correct above).

We'll recall that we said that, given:

$$\phi'' + \alpha(x)\phi' + \beta(x)\phi = 0$$

We can put this in Sturm-Liouville form by computing the integrating factor:

$$\mu(x) = e^{\int \alpha(x)\,dx}$$

Then, multiplying both sides by it, we have:

$$e^{\int \alpha(x)\,dx}(\phi'' + \alpha(x)\phi') = \left(e^{\int \alpha(x)\,dx}\phi'\right)'$$

So in this particular case, first we'll put in our standard form:

$$\phi'' - \frac{2x}{1-x^2}\phi' + \frac{1+\lambda x}{1-x^2}\phi = 0$$

The integrating factor is

$$\mu = e^{\int -2x/(1-x^2)\,dx} = 1-x^2$$

And therefore, the equation, in standard form, looks like:

$$((1-x^2)\phi')' + (1+\lambda x)\phi = 0 \quad \Rightarrow \quad ((1-x^2)\phi')' + \phi = -\lambda x\phi$$

I like to write the answer in eigenvalue form, but you could have left your expression without putting λ on the right.

9. Given the differential equation: $\phi'' + \lambda\phi = 0$, determine the eigenvalues λ and eigenfunctions ϕ if ϕ satisfies the following boundary conditions (analyze all three cases; you may assume the eigenvalues are real).

   (a) $\phi(a) = 0$, $\phi(b) = 0$

   NOTE: To solve this part, we need to use a trig identity:

   $$\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$

   It slipped past me as I was putting this together- You will NOT need to memorize this for the exam (although for other things, it wouldn't hurt).

   SOLUTION: The solutions to the characteristic equation are $r = \pm\sqrt{-\lambda}$.

   - Case 1: $\lambda = 0$. The solution is $\phi(x) = C_1 + C_2 x$. Using the boundary conditions, $C_1 = C_2 = 0$, and we have the trivial solution.
   - Case 2: $\lambda < 0$, or two real distinct solutions to the characteristic equation. In this case, writing the solutions in exponential form, we have

     $$\phi(x) = C_1 \mathrm{e}^{\sqrt{-\lambda}x} + C_2 \mathrm{e}^{-\sqrt{-\lambda}x}$$

     Using the boundary conditions, we have:

     $$\begin{aligned} C_1 \mathrm{e}^{\sqrt{-\lambda}a} + C_2 \mathrm{e}^{-\sqrt{-\lambda}a} &= 0 \\ C_1 \mathrm{e}^{\sqrt{-\lambda}b} + C_2 \mathrm{e}^{-\sqrt{-\lambda}b} &= 0 \end{aligned}$$

     These lines are not the same (unless $a = b$), so the only solution for this set is $C_1 = C_2 = 0$, and again we have a trivial solution.
   - Finally, if $\lambda > 0$, we have our usual solution:

     $$\phi(x) = A_n \cos\left(\sqrt{\lambda}x\right) + B_n \sin\left(\sqrt{\lambda}x\right)$$

     With the boundary conditions, we have:

     $$\begin{aligned} A_n \cos\left(\sqrt{\lambda}a\right) + B_n \sin\left(\sqrt{\lambda}a\right) &= 0 \\ A_n \cos\left(\sqrt{\lambda}b\right) + B_n \sin\left(\sqrt{\lambda}b\right) &= 0 \end{aligned}$$

     Therefore, we look for non-trivial solutions to this system. Using linear algebra and/or Cramer's rule, we know that there is a non-trivial solution if the determinant of the coefficient matrix is zero:

     $$\cos\left(\sqrt{\lambda}a\right)\sin\left(\sqrt{\lambda}b\right) - \cos\left(\sqrt{\lambda}b\right)\sin\left(\sqrt{\lambda}a\right) = 0$$

     or, if $\sin\left(\sqrt{\lambda}(b-a)\right) = 0$. Therefore, we have:

     $$\lambda_n = \left(\frac{n\pi}{b-a}\right)^2 \qquad \phi_n(x) = \sin\left(n\pi\frac{x-a}{b-a}\right)\sin\left(\sqrt{\lambda}a\right)$$

   (b) $\phi'(0) = 0$ and $\phi'(L) = 0$

   NOTE: This one and the next are fairly standard types of problems...

   SOLUTION: The solutions to the characteristic equation are $r = \pm\sqrt{-\lambda}$.

- Case 1: $\lambda = 0$. The solution is $\phi(x) = C_1 + C_2 x$. Using the boundary conditions, $\phi(x) = C_1$ is a possible solution.
- Case 2: $\lambda < 0$, or two real distinct solutions to the characteristic equation. In this case, writing the solutions in exponential form, we have

$$\phi(x) = C_1 \mathrm{e}^{\sqrt{-\lambda}x} + C_2 \mathrm{e}^{-\sqrt{-\lambda}x}$$

  Using the boundary conditions, we have:

$$\begin{aligned} \sqrt{-\lambda}C_1 - \sqrt{-\lambda}C_2 &= 0 \\ \sqrt{-\lambda}C_1 \mathrm{e}^{\sqrt{-\lambda}L} - \sqrt{-\lambda}C_2 \mathrm{e}^{-\sqrt{-\lambda}L} &= 0 \end{aligned}$$

  The only solution to this system is $C_1 = C_2 = 0$, so this has only the trivial solution.
- Finally, if $\lambda > 0$, we have our usual solution:

$$\phi(x) = A_n \cos\left(\sqrt{\lambda}x\right) + B_n \sin\left(\sqrt{\lambda}x\right)$$

  With the boundary conditions, we have:

$$\phi'(0) = 0 \quad \Rightarrow \quad \sqrt{\lambda}B_n = 0 \quad \Rightarrow \quad B_n = 0$$

$$\phi'(L) = 0 \quad \Rightarrow \quad -\sqrt{\lambda}A_n \sin\left(\sqrt{\lambda}L\right) = 0 \quad \Rightarrow \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

  and

$$\phi_n(x) = \cos\left(\frac{n\pi x}{L}\right)$$

  (And we don't want to forget $\lambda_0 = 1$ with $\phi_0(x) = 1$)

(c) $\phi(0) = 0$ and $\phi'(L) = 0$

SOLUTION: With the same solution to the characteristic equation, we have:

- $\lambda = 0$: $\phi(x) = C_1 + C_2 x$. With the two boundary conditions, $\phi(x) = 0$ is the only solution.
- $\lambda < 0$, and we get two distinct real solutions. Putting in the boundary conditions will yield a system for which the only solution is $C_1 = C_2 = 0$.
- The last case: $\lambda > 0$:

$$\phi_n(x) = A_n \cos(\sqrt{\lambda}\,x) + B_n \sin(\sqrt{\lambda}\,x)$$

  The first condition, $\phi(0) = 0$ makes $A_n = 0$, leaving only the sine expansion. The second condition is satisified if:

$$\sqrt{\lambda}B_n \cos(\sqrt{\lambda}\,L) = 0 \quad \Rightarrow \quad \sqrt{\lambda}L = \frac{2n-1}{2}\pi$$

  (That is, we need odd multiples of $\pi/2$ for the cosine). Therefore, we now have

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2 \qquad \phi_n(x) = \sin\left(\frac{(2n-1)\pi}{2L}x\right)$$

10. Solve

$$u_{tt} = \frac{4}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \qquad 0 < r < 1, t > 0$$

with $u(r,0) = f(r)$, $u(0,t)$ bounded and $u(1,t) = 0$. You should assume that the (radial) eigenfunctions are known and complete.

NOTE: We can leave 4 with ϕ or with T- In the solution below, we group it with T, but that isn't the only way to solve this.

SOLUTION: Using separation of variables with $u = \phi(r)T(t)$, we have:

$$\phi(r)T''(t) = \frac{4}{r}\left(r\phi(r)T(t)\right)_r \quad \Rightarrow \quad \frac{T''}{4T} = \frac{1}{\phi r}(r\phi'(r))_r = -\lambda$$

Therefore, dividing everything by $4\phi T$, we have:

$$(r\phi'(r))' = -\lambda r\phi \qquad T'' = -4\lambda T$$

The radial equation is in S-L form with $p(r) = r$, $q = 0$ and $\sigma(r) = r$. To solve the time equation, it is good to first see if we have any negative eigenvalues by checking the Rayleigh quotient:

$$\lambda_1 = R(\phi_1) = \frac{-r\phi\phi'|_0^1 + \int_0^1 r(\phi')^2\,dr}{\int_0^1 \phi^2 r\,dr}$$

From the boundary conditions, we know $\phi(1) = 0$ and $|\phi(r)|$ is bounded at $r = 0$. The Rayleigh quotient then simplifies to:

$$\lambda_1 = \frac{\int_0^1 r(\phi')^2\,dr}{\int_0^1 \phi^2 r\,dr}$$

For $\lambda_1 = 0$, we would require $\phi' = 0$, or $\phi(x) = C$. But $\phi(1) = 0$ would make this the trivial solution. Therefore, $\lambda_1 > 0$.

Now proceeding to the time equation with $\lambda > 0$:

$$T(t) = A_n\cos(\sqrt{\lambda}t) + B_n\sin(\sqrt{\lambda}t)$$

and the general solution is:

$$u(r,t) = \sum_{n=1}^{\infty}\phi_n(r)(A_n\cos(\sqrt{\lambda_n}t) + B_n\sin(\sqrt{\lambda_n}t))$$

with

$$u(r,0) = f(r) = \sum_{n=1}^{\infty}A_n\phi_n(r) \quad \Rightarrow \quad A_n = \frac{\int_0^1 f(r)\phi_n(r)r\,dr}{\int_0^1 \phi_n^2(r)r\,dr}$$

(Remember to multiply by r, which is required for orthogonality to hold).

11. Given the BVP in regular S-L form, with appropriate boundary conditions, show that the eigenfunctions corresponding to two distinct eigenvalues are orthogonal with respect to $\sigma(x)$. Hint: Consider

$$\int_a^b \phi_n L(\phi_m) - \phi_m L(\phi_n)\,dx$$

SOLUTION: Recall that for L to be the S-L operator, we have:

$$L(\phi) = -\lambda\sigma(x)\phi$$

Therefore,

$$\int_a^b \phi_n L(\phi_m) - \phi_m L(\phi_n)\,dx = \int_a^b -\lambda_m\phi_n\phi_m\sigma(x) + \lambda_n\phi_m\phi_n\sigma(x)\,dx$$

$$= (\lambda_n - \lambda_m) \int_a^b \phi_n \phi_m \sigma(x)\, dx$$

We also know (proof using Green's formula) that if u, v satisfy the (regular) boundary conditions, then

$$\int_a^b uL(v) - vL(u)\, dx = 0$$

And our ϕ_n, ϕ_m do indeed satisfy the BCs. Therefore, we conclude that

$$(\lambda_n - \lambda_m) \int_a^b \phi_n \phi_m \sigma(x)\, dx = 0$$

Since $\lambda_n \neq \lambda_m$, the integral must be zero (for any $n \neq m$).

12. Solve using separation of variables:

$$\begin{array}{ll} \text{PDE} & u_{tt} = u_{xx} \qquad 0 < x < 1, t > 0 \\ \text{BCs} & u(0,t) = 0 \quad u(1,t) = 0 \\ \text{ICs} & u(x,0) = \sin(\pi x) + \frac{1}{2}\sin(3\pi x) + 3\sin(7\pi x) \\ & u_t(x,0) = \sin(2\pi x) \end{array}$$

(Keep the constants with the spatial equation)

NOTE: In this case, there weren't any constants to keep with the spatial equation- That was a copy-paste error, so ignore it.

SOLUTION: We recognize that we'll get the standard equations for space and time, with the usual boundary conditions for the sine expansion. That is:

$$T'' = -\lambda T \qquad \phi'' = -\lambda \phi$$

with

$$\lambda_n = (n\pi)^2 \qquad \phi_n(x) = \sin(n\pi x)$$

We won't have any zero eigenvalues, so we can also solve the temporal equation right away:

$$T_n(t) = A_n \cos(n\pi t) + B_n \sin(n\pi t)$$

so the overall solution is:

$$u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)(A_n \cos(n\pi t) + B_n \sin(n\pi t))$$

Now,

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) = \sin(\pi x) + \frac{1}{2}\sin(3\pi x) + 3\sin(7\pi x)$$

Therefore, $A_1 = 1, A_3 = 1/2, A_7 = 3$ and all other A_n are zero. Using the other initial condition,

$$u_t(x,0) = \sum_{n=1}^{\infty} n\pi B_n \sin(n\pi x) = \sin(2\pi x)$$

Therefore, all $B_n = 0$ except for B_2. In this case, $B_2 = frac12\pi$.

Putting all the pieces together, the solution is:

$$u(x,t) = \cos(\pi t)\sin(\pi x) + \frac{1}{2}\cos(3\pi t)\sin(3\pi x) + 3\cos(7\pi t)\sin(7\pi x) + \frac{1}{2\pi}\sin(2\pi t)\sin(2\pi x)$$

13. Consider $L(\phi) = \phi''''$. Find an expression like Green's Formula for this operator on $0 < x < 1$. HINT: Use integration by parts for $\int_0^1 uL(v)\,dx$ until you get $\int_0^1 vL(u)\,dx$ on the right side of the equation. That is, you should have something in the form:

$$\int_0^1 uL(v)\,dx = (...) + \int_0^1 vL(u)\,dx$$

SOLUTION: Using integration by parts with a table, we have the following:

$$\int_0^1 uL(v)\,dx \quad \Rightarrow \quad \begin{array}{ccc} + & u & v'''' \\ - & u' & v''' \\ + & u'' & v'' \\ - & u''' & v' \\ + & u'''' & v \end{array}$$

Putting this together,

$$\int_0^1 uL(v)\,dx = (uv''' - u'v'' + u''v' - u'''v)|_0^1 + \int_0^1 vu''''\,dx$$

Therefore,

$$\int_0^1 uL(v) - vL(u)\,dx = (uv''' - u'v'' + u''v' - u'''v)|_0^1$$

14. Consider

$$\begin{aligned} \rho u_{tt} &= T_0 u_{xx} + \alpha u \\ u(0,t) = u(L,t) &= 0 \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}$$

with $\rho(x) > 0$, $\alpha(x) < 0$, and T_0 constant.

Assume that the appropriate eigenfunctions (in space) are known. Solve the PDE using separation of variables.

SOLUTION: See 5.4.5, included below.

Separation of variables will lead to our usual two ODEs, one in time, one in space:

$$\begin{aligned} \text{(time) } h'' + \lambda h &= 0 \\ \text{(space) } \phi'' + \frac{\alpha}{T_0}\phi + \lambda\frac{\rho}{T_0}\phi &= 0 \end{aligned}$$

The spatial equation is a regular Sturm-Liouville problem with $p(x) = T_0$ (constant), $q(x) = \alpha < 0$ and $\sigma(x) = \rho(x) > 0$. The Rayleigh quotient then becomes:

$$\lambda = \frac{0 + \int_0^L T_0(\phi')^2 - \alpha\phi^2\,dx}{\int_0^L \phi^2\,\rho\,dx}$$

Since $\alpha < 0$, this quotient is greater than zero (zero only when $\phi = 0$). Therefore, we know $\lambda_n > 0$. Therefore, the solutions to the time equation are:

$$h_n(t) \quad = \quad c_n\cos(\sqrt{\lambda_n}t) + d_n\sin(\sqrt{\lambda_n}t)$$

Using the superposition, we have

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} \phi_n(x)[a_n \cos(\sqrt{\lambda_n}t) + b_n \sin(\sqrt{\lambda_n}t)] \\
u(0,t) &= \sum_{n=1}^{\infty} a_n \phi_n(x) = f(x) \\
a_n &= \frac{(f, \sigma\phi_n)}{(\phi_n, \sigma\phi_n)}
\end{aligned}$$

where $\sigma = \rho/T_0$, and (f,g) is the inner product

$$(f,g) = \int_0^L f(x)g(x)\,dx$$

Also, for initial velocity

$$\begin{aligned}
u_t(x,t) &= \sum_{n=1}^{\infty} \phi_n(x)[-a_n\sqrt{\lambda_n} \sin(\sqrt{\lambda_n}t) + b_n\sqrt{\lambda_n} \cos(\sqrt{\lambda_n}t)] \\
u_t(0,t) &= \sum_{n=1}^{\infty} b_n\sqrt{\lambda_n}\phi_n(x) = g(x) \\
b_n &= \frac{1}{\sqrt{\lambda_n}} \frac{(g, \sigma\phi_n)}{(\phi_n, \sigma\phi_n)}
\end{aligned}$$

15. Use the Rayleigh quotient to obtain a reasonably accurate upper bound for the lowest eigenvalue of

$$\phi'' + (\lambda - x)\phi = 0 \qquad \phi'(0) = 0 \quad \phi'(1) + 2\phi(1) = 0$$

SOLUTION: Find the simplest (nonzero!) function that satisfies the boundary conditions. A constant or a line will be trivial, so the next best thing is a polynomial of degree 2:

$$u(x) = ax^2 + bx + c \quad u'(0) = 0 \quad \Rightarrow b = 0$$

Now, with $\phi'(1) + 2\phi(1) = 0$, we have: $(2a) + 2(a+c) = 0$ or $4a + 2b = 0$, or $b = -2a$. For example, we can try

$$u = x^2 - 2$$

Now, with $p(x) = 1$, $\sigma(x) = 1$, and $q(x) = -x$, the Rayleigh quotient is:

$$\lambda \leq \frac{0 + \int_0^1 (2x)^2 + x(x^2-2)^2\,dx}{\int_0^1 (x^2-2)^2\,dx} \approx 0.87$$

16. Use the alternate form of the Rayleigh quotient below to compute $R(u)$, if λ_n are the eigenvalues, ϕ_n the eigenfunctions, and $u = 2\phi_1 + 3\phi_2$. To simplify our computations, you may assume that the eigenfunctions have been normalized so that $\int_a^b \phi_n^2 \sigma(x)\,dx = 1$. In that case, the Rayleigh quotient simplifies to:

$$R(\phi) = -\int_a^b \phi L(\phi)\,dx$$

NOTE: This exercise is to familiarize you with the technique we used in proving the minimization principle...

SOLUTION:

$$u = 2\phi_1 + 3\phi_2 \quad \Rightarrow \quad L(u) = -2\lambda_1\sigma(x)\phi_1 - 3\lambda_2\sigma(x)\phi_2$$

so that when we multiply $uL(u)$, we will have mixed terms $\phi_i\phi_j$, so we integrate to make them zero, and we're left with:

$$-\int_a^b uL(u)\,du = 2^2\lambda_1^2 \int_a^b \phi_1^2\sigma(x)\,dx + 3^2\lambda_2^2 \int_a^b \phi_2^2\sigma(x)\,dx = 4\lambda_1^2 + 9\lambda_2^2$$

Similarly,

$$\int_a^b u^2\sigma(x)\,dx = 2^2 \int_a^b \phi_1^2\sigma(x)\,dx + 3^2 \int_a^b \phi_2^2\sigma(x)\,dx = 2^2 + 3^2 = 13$$

Therefore,

$$R(u) = \frac{4}{13}\lambda_1 + \frac{9}{13}\lambda_2$$

17. Suppose we define a linear operator as: $L(y) = y'$, where y satisfies the BCs $y(0) - 3y(1) = 0$. Find an expression for the adjoint operator L^* so that

$$\langle u, L(v)\rangle = \langle L^*(u), v\rangle$$

You may assume that $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$. You should also determine the BCs for functions used by L^*.

SOLUTION: We write out $\langle u, L(v)\rangle$, then we do something so that v factors out, leaving an expression in u (typically this means integration by parts):

$$\int_0^1 uv'\,dx \quad \Rightarrow \quad \begin{array}{ccc} + & u & v' \\ - & u' & v \end{array} \quad \Rightarrow \quad uv|_0^1 - \int_0^1 vu'\,dx$$

Work with the constant term, and note that u satisfies the BCs given in the problem (but v may not):

$$u(1)v(1) - u(0)v(0) = u(1)v(1) - 3u(1)v(0) = u(1)(v(1) - 3v(0))$$

Therefore, we can say that $L^*(v) = -v'$ and the BC is $v(1) - 3v(0) = 0$ (which has an interesting relationship to the original BC). As a side remark, we note that $L \neq L^*$ (and the domains are also different), so differentiation is not self-adjoint (but the second derivative is).