Extra Practice:
Solve the Wave Equation with "Free Ends"

The PDE we consider is:

$$\begin{array}{rll} \text{PDE} & u_{tt} = c^2 u_{xx} & 0 < x < L \\ \text{BCs} & u_x(0,t) = 0, & u_x(L,t) = 0 \\ \text{ICs} & u(x,0) = f(x) & u_t(x,0) = g(x) \end{array}$$

We'll solve using separation of variables:

$$u = XT \quad \Rightarrow \quad XT'' = c^2 X'' T \quad \Rightarrow \quad \frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda$$

Therefore, splitting the ODEs up, we have:

$T'' + \lambda c^2 T = 0$		$X'' + \lambda X = 0$ $X'(0) = 0 = X'(L)$
$T(t) = a_0 + b_0 t$	$\lambda = 0$	$X(x) = C_1$
	$\lambda < 0$	$X(x) = C_1 \mathrm{e}^{\sqrt{-\lambda}\,x} + C_2 \mathrm{e}^{-\sqrt{-\lambda}\,x}$ $X'(0) = X'(L) = 0 \quad \Rightarrow \quad X(x) = 0$
	$\lambda > 0$	$X(x) = C_1 \cos(\sqrt{\lambda}\,x) + C_2 \sin(\sqrt{\lambda}\,x)$

Finishing up this last solution in X, T, we see that for X,

$$X'(0) = 0 \quad \Rightarrow \quad \sqrt{\lambda} C_2 = 0 \quad \Rightarrow \quad C_2 = 0$$

For the other end, for $n = 1, 2, 3, \cdots$,

$$X'(L) = 0 \quad \Rightarrow \quad -\sqrt{\lambda} C_1 \sin(\sqrt{\lambda}\,L) = 0 \quad \Rightarrow \quad \sqrt{\lambda} L = n\pi$$

Therefore, the eigenvalues and eigenfunctions are:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad X_n(x) = \cos\left(\frac{n\pi}{L}\,x\right)$$

For time, we have:

$$T'' + \left(\frac{n\pi c}{L}\right)^2 T = 0$$

so that

$$T(t) = A_n \cos\left(\frac{n\pi c}{L}\,t\right) + B_n \sin\left(\frac{n\pi c}{L}\,t\right)$$

The general solution is:

$$u(x,t) = a_0 + b_0 t + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{L}\,x\right)\left(A_n \cos\left(\frac{n\pi c}{L}\,t\right) + B_n \sin\left(\frac{n\pi c}{L}\,t\right)\right)$$

To solve for the coefficients, we go back to the initial conditions:

$$u(x,0) = a_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}\,x\right) = f(x)$$

Therefore, these coefficients are the Fourier cosine coefficients for $f(x)$:

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx \qquad A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}\,x\right)\,dx,\ \text{ for } n = 1, 2, 3, \cdots$$

Similarly,

$$u_t(x,0) = b_0 + \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L}\cos\left(\frac{n\pi}{L}\,x\right) = g(x)$$

Therefore,

$$b_0 = \frac{1}{L}\int_0^L g(x)\,dx \qquad B_n\frac{n\pi c}{L} = \frac{2}{L}\int_0^L g(x)\cos\left(\frac{n\pi}{L}\,x\right)\,dx,\ \text{ for } n = 1, 2, 3, \cdots$$